INTERIOR BALLISTICS
1887

MEIGS—INGERSOLL


ERECTUS NON ELATUS

INTERIOR BALLISTICS;

A TEXT BOOK

FOR THE USE OF CADETS AT THE U. S. NAVAL ACADEMY.

BY

ohn orsyth
LIEUTENANT J. F. MEIGS, U. S. N.

AND

LIEUTENANT R. R. INGERSOLL, U. S. N.

1887.

PRESS AND BINDERY OF
ISAAC FRIEDENWALD,
BALTIMORE, MD.

Engineering Library
From the Library of
George Owen Squier
5-3-45

CONTENTS.

PREFACE.

In 1884, Sarrau's Researches on the Effects of Powder were translated and published in the Proceedings of the U. S. Naval Institute, Vol. X, No. 28, by the authors of this volume, and in that form were used in the Department of Ordnance and Gunnery at the Naval Academy for the instruction of Cadets. The edition having been exhausted, it has been thought proper to reproduce those portions of Sarrau's researches which are necessary for a practical understanding of the work of gunpowder in guns. Numerous examples have been introduced for work in the Section Room, and some extracts from Sarrau's Chargement des Bouches à Feu have been added.

ANNAPOLIS, MD., *August*, 1887.

CHAPTER I.

1. Definitions.—*Interior Ballistics*, in general terms, may be said to include the effects produced on a projectile in the bore of a gun, and on the gun itself, when the gun and projectile are subjected to the action of the products of combustion of gunpowder or other explosives. Primarily the object of the study of the subject is to ascertain the velocity of translation and rotation at the muzzle which a given charge imparts to a projectile, and the maximum pressure produced on the walls of the gun. There are, however, many other questions involved of scarcely less importance, such as the effect on velocity and pressure of the different elements of loading, the laws of combustion of gunpowder in grains of varied form in free air, the effect of explosives when fired in a closed receptacle, etc., etc., all of which will be treated of in their appropriate places.

Some of the terms which will be frequently met with in subsequent pages will now be defined.

2. The Density of Gunpowder.—The *specific gravity* of gunpowder, frequently called the *density* of the powder, is the ratio of the weight of a grain of gunpowder to the weight of an equal volume of water in standard conditions. In practice the mean specific gravity of a number of grains must be determined. The practical limits of the density are from 1.68 to 1.90, in fact rarely exceeding 1.85.

3. Gravimetric Density.—The gravimetric density of gunpowder is the weight of a standard volume when in the grain form, generally from 875 to 975 ounces per cubic foot.

4. Density of Loading.—The density of loading is the ratio of the weight of the charge to the capacity of the powder chamber or receptacle in which the charge is placed, when expressed in comparable units such as weight of charge in grams and capacity of chamber in cubic centimetres, or weight of charge in kilograms and capacity in cubic decimetres. It may also be defined as the ratio of the weight of the charge to the weight of a volume of water equal to the whole capacity of the powder chamber. In French units the

weight of a cubic centimetre of water being one gram, the comparison is best made in those units.

5. The phenomenon of the explosion of gunpowder may be divided into three parts, viz. ignition, inflammation, and combustion, and these should not be confounded with each other.

6. Ignition.—By ignition is understood the setting on fire of a particular part of the charge. With B. L. modern guns this is effected at the rear of the charge; and to insure ignition, a small priming charge of black powder is sometimes placed at the rear of heavy charges of cocoa powder.

7. Inflammation.—By inflammation is meant the spreading of the fire from grain to grain throughout the whole charge. The small priming charge spoken of in the preceding definition assists in this operation in the case of cocoa powder, which ignites somewhat slowly.

8. Combustion.—By combustion is meant the burning of *each* grain from its surface to its centre.

By *velocity of inflammation*, then, we mean the rate at which the heated and expansive gases evolved from the first grain ignited insinuate themselves into the interstices of the charge, envelop the grains, and ignite them one after another.

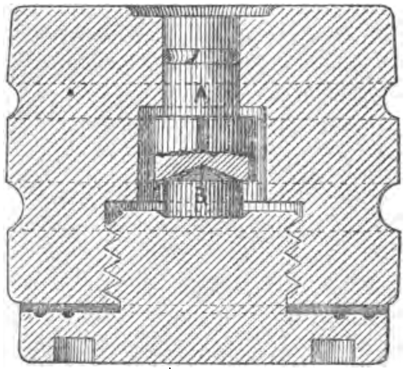
By *velocity of combustion* we mean the rate at which the grain burns from its surface to its centre. It is important to keep in mind the difference of meaning of these two terms when we come to the mathematical deductions of the laws of burning of powder grains in free air.

Proof of Gunpowder.—When gunpowder arrives at the proof-grounds for trial in the gun for which it is intended, it is subjected to examination as to its specific gravity, gravimetric density, and its pressure and velocity in the gun are ascertained with a pressure gauge and a chronoscope.

Since nitre is very soluble in water, the practice of weighing gunpowder in mercury of known specific gravity has been universally adopted. In the instruments used, every refinement is adopted to secure precision. The gravimetric density is ascertained by filling a standard volume and carefully weighing the contents.

Firing Test.—Powder is tested by firing a charge in the gun for which it is intended, pressure gauges being in the breech plug, base

of the shell, or elsewhere in the bore, while the velocity of the projectile is determined by a chronoscope. The cut represents a Woodbridge pressure gauge. The piston A is forced down by the powder pressure and indents the copper disk B, forming a cone on which is traced a helix. The number of convolutions of the helix which appear on the disk shows, by comparison with disks which have been equally indented by known pressures, the pressure reached in the



gun. Of course, in practice, the first step in the use of the gauge is to determine a number of points and draw a curve; the abscissæ being, say, the convolutions of the helix, and the ordinates the pressures. The other well known pressure gauges are the Rodman and the Crusher gauges. In the former, the pressures are ascertained by the indentation of a knife, and in the latter by the shortening of a metal cylinder.

The Chronoscope used on most firing grounds is the Boulengé; this consists essentially of a metallic rod, which is held by an electro-magnet at its upper end. A projectile, by cutting the wires in a screen, ruptures the current in the electro-magnet, and the rod begins to fall; when the projectile cuts the wires of a second screen, a knife flies forward and marks the rod. Thus the time during which the rod was falling becomes known, and this, when properly corrected, is the time the shot took between screens. There are other well known chronoscopes, such as the Schultz, Noble and Bashforth. The first depends upon the fact that the number of vibrations of a tuning fork in the same temperature is constant, and the second upon the attainment of a very high velocity of a recording surface. In all these instruments electricity is used for purposes of recording.

The Ballistic Effect.—The ballistic effect reached depends, per haps, more upon the powder, and the manner of its use, than upon anything else. The study of this subject is involved in great difficulty, from the large number of variables which it presents; but, from its extreme importance, it cannot be neglected. The energy of the shot at the muzzle,

$$\frac{WV^2}{2g},$$

it is often asserted, is the measure of the ballistic effect; but it is evident that the artillerist cannot stop there, for the effect may have been reached by a pressure which has endangered the gun, by an inconvenient length of bore or weight of gun, or by burning a wasteful amount of powder.

In statements of ballistic effect of any gun, it is usual to give the values of the quantities muzzle energy per ton of gun, per ton of pressure, per pound of powder burned, per inch of shot's circumference, and per square inch of cross section. The last two are considered the measure of the shot's penetrative effect, according to which of two theories is adopted. It is evident that of two shots of equal weight and velocity, the one which is longer, with the lesser radius, will give the greatest value to each of these.

By the study of the first three the degree of excellence of the gun and its charge may be ascertained by comparison with other practice. It has already been stated that, if powder could be made to burn as we please, the value of the quantity $\frac{\text{Muzzle energy}}{\text{Gun's weight}}$ can be increased by making the gun a cylinder without increasing either its length or weight, and it is easy to show that we may further increase it by lengthening the cylinder, while still keeping the weight the same. Thus we shall attain its greatest value by making the gun a very long thin cylinder. The limits in this direction are, however, set by soon reaching too great a length of gun, and by the possibilities of powder manufacture. Closely connected with the quantity $\frac{\text{Muzzle energy}}{\text{Gun's weight}}$ is $\frac{\text{Muzzle energy}}{\text{Pressure}}$; for, in the case just imagined of a very long thin cylinder, with a low constant powder pressure acting through its length, these would both reach their maximum. The second has its significance and clearness greatly increased by reckoning the whole pressure on the base of the projectile, instead of the pressure per

unit area, as is customary. For, in the latter case, it increases, other things being equal, inversely as the square of the calibre; if, however, it be reckoned in the manner just mentioned, its dimensions are those of a length, and in our imaginary case it becomes the length of the gun. In actual guns the ratio of this length to the length of the gun is an exact measure of the progressiveness of the powder: perfect progressiveness being that of the imaginary case cited, and being represented numerically by unity. Thus reckoned, we may also compare different calibres.

We would reach, then, a large value of the quantities $\frac{\text{Energy}}{\text{Gun's weight}}$ and $\frac{\text{Energy}}{\text{Pressure}}$ by causing the gun to approach in form to a long cylinder, and by causing the powder to burn so as to keep up a constant low pressure. But what change would $\frac{\text{Energy}}{\text{Powder weight}}$ then undergo? It is clear that, since when the base of the projectile reaches the muzzle the powder gas is still expanding (and in an actual case has still a very high pressure), we shall waste a very large part of the energy of the gas. Thus, any gun and charge which have been designed to attain, and have attained, large values of these three quantities, constitute a compromise, and the limit of excellence has not been reached in any direction.

Let us now trace the inverse problem, and try to reach great economy in expenditure of powder. We must assume something about its behavior, and, if we suppose that it is instantly converted into a perfect gas, before the shot moves, and then expands without gain or loss of heat; then, to reach a maximum, the gun must be a very thick cylinder over the chamber, forward of that it must thin very rapidly, and it must be infinite in length. An approach to this form will, of course, cause a corresponding increase in the quantity we are considering. The numerical value reached for any number of expansions up to 20 can be found in the table on p. 165, Mackinlay; the value reached in the case imagined, of indefinite expansion, is 486 foot-tons per pound of powder (Noble and Abel). It is obvious than an actual gun whose form approaches that stated would give small values to $\frac{\text{Energy}}{\text{Gun's weight}}$ and $\frac{\text{Energy}}{\text{Pressure}}$.

Thus the attempt to reach economy in the use of powder will cause certainly a diminution of $\frac{\text{Energy}}{\text{Pressure}}$, and, in any actual case, a diminu-

tion of $\frac{\text{Energy}}{\text{Gun's weight}}$ as well. Since the limit of safe chamber pressure in guns cannot now be materially increased, and since the pressure falls off so notably towards the muzzle, the endeavor at present among gun designers is to increase progressiveness in the powder; this of course entails a change in the external form of the gun.

If we call the weights of the gun, projectile, and charge W_1, W_2, and W_3 respectively, we have

$$\frac{\text{Energy}}{\text{Gun's weight}} = \frac{\frac{W_2 V^2}{2g}}{W_1} = \frac{W_2}{W_1} h,$$

$$\frac{\text{Energy}}{\text{Total pressure}} = \frac{\frac{W_2 V^2}{2g}}{P} = \text{“effective” length},$$

$$\frac{\text{Energy}}{\text{No. lbs. powder}} = \frac{\frac{W_2 V^2}{2g}}{W_3} = \frac{W_2}{W_3} h;$$

where h is the height which would produce the velocity V in a body falling in a vacuum. In the ordinary conditions of practice with heavy charges, $\frac{W_2}{W_3} = \frac{2}{1}$ and $\frac{W_2}{W_1} = \frac{1}{105}$, and in this case we have

Energy per pound of powder $=$ 210 $\times$ Energy per pound of gun.

The above discussion may be summarized thus: In ordinary cases we have three data respecting the pressure curve—1st, its area, $\frac{WV^2}{2g}$; 2d, its length, the distance from base of shot to face of muzzle; 3d, its maximum ordinate. If we draw a rectangle of equal area on the same base, its height will be the mean of all the pressures; and evidently $\frac{\text{mean pressure}}{\text{maximum pressure}} = \frac{\text{“effective” length}}{\text{travel of projectile}}$.

EXAMPLES.

1. The volume of the powder chamber of the 8-inch B. L. R. mark II is 3824 cu. in.; what is the *density of loading* when the charge is 125 pounds? *Ans.* .905.

2. What is the density of loading of the 8-inch B. L. R. mark II when the charge is 110 pounds? *Ans.* .798.

3. A 6-inch B. L. R. with a powder chamber of 1100 cu. in. capacity is loaded with a charge weighing 43 pounds; what is the density of loading? *Ans.* 1.082.

4. A 6-inch B. L. R., capacity of powder chamber 1426 cu. in., is loaded with 54 pounds of powder; what is the density of loading?

Ans. 1.048.

5. The density of loading for the 60-pounder B. L. R. with a 10-pound charge is .8987; what is the capacity of the powder chamber?

Ans. 308 cu. in.

6. The volume of the powder chamber of the South Boston 6-inch B. L. R. is 920 cu. in., and the density of loading with a certain charge was .752; what was the weight of charge used, to the nearest whole pound?

Ans. 32 pounds.

CHAPTER II.

PROPERTIES OF GASES.

9. **Mariotte's and Gay-Lussac's Laws.**—Gases and superheated vapors tend toward a limiting state, called that of a perfect gas, which is characterized by the two following laws:

10. **Mariotte's Law.**—The pressures of the same mass of gas are inversely proportional to the volumes.

11. **Gay-Lussac's Law.**—All gases have the same coefficient of dilatation under constant pressure, and this coefficient is independent of the pressure.

These two laws are expressed by the equation

$$pv = K(1 + at); \qquad (1)$$

where v is the volume of unit of weight of the gas, p the pressure, t the temperature, a the coefficient of dilatation under constant pressure, and K a constant depending upon the nature of the gas.

12. **Absolute Temperature.**—The coefficient of dilatation a, which is the same for all perfect gases according to Gay-Lussac's law, is $\frac{1}{273}$, nearly, according to the experiments of Regnault. Consequently equation (1) becomes

$$pv = \frac{K}{273}(273 + t). \qquad (2)$$

The factor $(273 + t)$ is called the absolute temperature in the dynamic theory of heat. It is the centigrade temperature counted from a zero placed 273 degrees below the ordinary zero.

13. **Specific Volume.**—If we call p_0 the normal atmospheric pressure, and put $K = p_0v_0$, equation (2) may be written

$$pv = \frac{p_0v_0}{273}(273 + t), \qquad (3)$$

and, in this form, we see that v_0 is the value of v which corresponds to the values $t = 0$ and $p = p_0$ of the temperature and pressure.

This constant, which is called the specific volume, represents, then,

the volume of unit of weight of the gas at zero temperature and under the normal pressure. If the gas under consideration can reach these conditions of pressure and temperature without change of state and without ceasing to satisfy Mariotte's and Gay-Lussac's laws, its numerical value is the reciprocal of the specific weight, or weight of the unit of volume at zero and under the pressure p_0.

The specific volume ceases evidently to have this signification for gases and vapors which are in the state of a perfect gas only at temperatures above zero. Its definition results from equation (3), which the gas under consideration must satisfy in the limiting state; its numerical value may be derived from that equation by determining by experiment, in the physical conditions where the equation is applicable, a system of values corresponding to p, v, and t.

The determination of specific volumes, or, what is the same thing, the densities of gases and vapors, is of great importance in chemistry, and they have been the object of the research of many eminent scientists. We shall return, at the end of this chapter, to the physical laws which have been established, dwelling particularly upon those which are useful in the approximate calculation of the force of explosive substances.

14. Specific Heats.—The specific heat of a substance is the quantity of heat necessary to raise unit of weight of the substance one degree centigrade. The quantity may be measured in two ways; the body which is being heated may be allowed to expand freely under a determined pressure, or the volume of the body may be maintained constant; in the first case the specific heat is of constant pressure, and in the second of constant volume.

Calorimetric experiments have established the following law, which, like those of Mariotte and Gay-Lussac, appears to characterize the state of the perfect gas:

The specific heats under constant pressure and constant volume are independent of the pressure and volume.

This law has been experimentally verified by Regnault, by the direct determination of specific heats under constant pressure.

As to the specific heats for constant volume, their direct measurement being almost impossible, this verification cannot be made. The law appears, however, sufficiently confirmed by the indirect measurement which has been made of the relation between the specific heats throughout a considerable range of temperature and pressure, by many investigators.

15. Conversion of Heat into Work by the Expansion of Gas.—The above laws being established, we may deduce as follows the relation that exists between the variations of volume and pressure of a gas and the heat necessary to produce them.

Resuming the fundamental equation (3), and writing for shortness

$$R = \frac{p_0 v_0}{273}, \tag{4}$$

and calling T the absolute temperature $(273 + t)$, this equation may be written

$$pv = RT. \tag{5}$$

The consequences which follow are these:

1st. If, the pressure p remaining constant, the volume varies by the amount dv, the temperature undergoes a corresponding variation, represented by $\frac{pdv}{R}$; and, consequently, the gas has received a quantity of heat $\frac{c'pdv}{R}$, c' being the specific heat under constant pressure.

2d. If, the volume v remaining constant, the pressure varies by dp, the temperature varies by $\frac{vdp}{R}$, and the gas has received a quantity of heat equal to $\frac{cvdp}{R}$, c being the specific heat under constant volume.

3d. Consequently, if the volume and pressure increase together by dv and dp, the gas receives a quantity of heat

$$dq = \frac{1}{R}(c'pdv + cvdp). \tag{6}$$

Differentiating (5), we have

$$RdT = pdv + vdp, \tag{7}$$

and, eliminating successively between (6) and (7) dp and dv, we have the two equations

$$dq = cdT + \frac{c' - c}{R} pdv, \tag{8}$$

$$dq = c'dT - \frac{c' - c}{R} vdp, \tag{9}$$

which, together with (6), contain all the thermodynamic laws of gases.

16. We will consider first equation (8). The quantity pdv, which appears in its second member, is evidently the work done by the elastic force of the gas when the volume increases by dv.

Let ω be an infinitely small element of the surface which incloses the gas. The pressure on this element is $p\omega$, since p is the pressure on unit of surface; and the element of work of this pressure corresponding to an infinitely small increase of volume is $p\omega h$, h being the displacement of ω perpendicular to itself. The work done then is $p\Sigma\omega h$; but $\Sigma\omega h$ is the variation of the volume dv; therefore pdv is the external work done by the gas.

It results, then, from equation (8) that the quantity of heat absorbed by a gas in an infinitely small change is composed of two terms, of which the first is proportional to the change of temperature, and the second to the element of external work.

If we consider a change which alters by finite quantities the temperature and volume of a gas, we derive, by integrating (8), the quantity of heat absorbed during the change,

$$q\int\left(c\,dT+\frac{c'-c}{R}p\,dv\right).$$

If c and c' are functions of T and v, this expression can only be integrated if we know the relation between them; but, if we assume (No. 4) that c and c' are constants, we have

$$q=c\int dT+\frac{c'-c}{R}\int p\,dv,$$

or, calling T_0 and T_1 the initial and final temperatures, and 6 the total external work done by the elastic force of the gas, we have

$$q=c\,(T_1-T_0)+\frac{c'-c}{R}6. \qquad (10)$$

17. Equivalence of Heat and Work.—If we put $T_1=T_0$ in the last equation; that is, if the gas returns to its initial state, we have

$$q=\frac{c'-c}{R}6. \qquad (11)$$

Thus, in this case, the amount of heat absorbed by the gas is proportional to the external work done.

The quantity

$$A=\frac{c'-c}{R}, \qquad (12)$$

which expresses the ratio between the heat absorbed and the work done, is called the calorific equivalent of the work; and its reciprocal,

$$E=\frac{R}{c'-c}, \qquad (13)$$

is the mechanical equivalent of the heat.

We thus see that we may deduce from the known laws which govern gases the notion of the equivalence of heat and work, whose precise conception has led to so great progress in the theory of heat. The fundamental postulate of this theory consists in the assertion that the quantity which has been designated E is invariable, and *independent of the nature of the body* which has served as intermediate in the transformation of heat into work. We need not recall here the reasoning which has established the soundness of this principle, which may otherwise be considered as sufficiently confirmed by the verification of its numerous consequences.

We consider, then, that, for perfect gases, there exists between the volume and the two specific heats the relation expressed by (13); or, taking into consideration the value of R (4), the relation

$$E = \frac{1}{273} \frac{p_0 v_0}{c' - c} \tag{14}$$

expresses an invariable number, depending solely upon the choice of the units which serve to express the quantities of work and heat.

18. Value of the Mechanical Equivalent of Heat.—Since the value of E is independent of the nature of the gas, we may derive it from the values of v_0, c', and c, which have been experimentally determined for many gases, following the laws of Mariotte and Gay-Lussac. We take, for example, atmospheric air.

Regnault's experiments show that under constant pressure the specific heat of air is

$$c' = .23754,$$

and that its specific volume (the reciprocal of the specific weight), taking for units the metre and the kilogram, is

$$v_0 = \frac{1}{1.2932}.$$

Finally, the ratio between the two specific heats, deduced from the velocity of sound as observed by Regnault, is

$$\frac{c'}{c} = 1.3945,$$

from which we derive, for the specific heat of constant volume,

$$c = .17034.$$

Inserting these numerical values in (14), and putting $p_0 = 10333$, we find

$$E = 436. \tag{15}$$

This, then, is the value of the mechanical equivalent of heat, as determined from the best determined constants which are now obtainable. It is the value which we shall adopt. We must not lose sight of the fact, however, that there is some uncertainty as to its exact value, in consequence of the large result that small variations in the value of the ratio $\frac{c'}{c}$ would produce.

19. Adiabatic Transformations.—Up to this point we have considered only those transformations in the state of a gas which are caused by its losing a quantity of heat. It may, however, happen that the transformation takes place within an envelope which is impermeable to heat; in which case the gas may change its volume and pressure, and consequently its temperature, without gain or loss of heat. The change is then said to be adiabatic, and the laws which govern it may be derived from equations (6), (8), and (9), by putting $dq = 0$. We thus obtain the relations

$$c'pdv + cvdp = 0, \tag{16}$$

$$cdT + \frac{c' - c}{R} pdv = 0, \tag{17}$$

$$c'dT - \frac{c' - c}{R} vdp = 0, \tag{18}$$

from which follow several important results.

20. Law of Pressures.—Taking equation (16), and putting $n = \frac{c'}{c}$, it may be written

$$n\frac{dv}{v} + \frac{dp}{p} = 0,$$

whence, integrating, and calling f a constant,

$$n \log v + \log p = \log f,$$

or

$$v^n p = f. \tag{19}$$

Also v, the volume of the unit of weight, is the reciprocal of ρ, the weight of the unit of volume. Thus equation (19) may be written $p = f\rho^n$; or, in an adiabatic change, the pressure varies proportionally to a power of the density equal to the ratio of the two specific heats.

Such is the law of pressures discovered by Laplace and Poisson before the birth of the mechanical theory of heat.

21. Law of Temperatures.—Let us consider equation (17). If we put $c' = nc$, and for p its value $\frac{RT}{v}$, derived from (5), it becomes

$$\frac{dT}{T} + (n - 1)\frac{dv}{v} = 0,$$

whence, by integration, and calling f_1 a constant,

$$Tv^{n-1} = f_1. \tag{20}$$

Consequently, recollecting that, as above, v is the reciprocal of the density, we have this law: *The absolute temperature of a gas, in any adiabatic transformation, is proportional to a power of the density equal to the ratio of the two specific heats minus one.*

We may also present this law under another form. Equation (17) may be written

$$EcdT + pdv = 0,$$

by recollecting that we have, from (13), $\frac{c' - c}{R} = \frac{1}{E}$. Consequently, integrating, and calling T_0 and T_1 the initial and final temperatures of the gas, we have

$$Ec(T_0 - T_1) = \int pdv. \tag{21}$$

But the second member represents the total work done by the elastic force of the gas (No. 5). Thus, in an adiabatic transformation, *the temperature is lowered by a quantity which is proportional to the external work done.*

22. The Work done by the Indefinite Adiabatic Expansion of Gas.—If we suppose that the gas expands indefinitely without gain or loss of heat, the volume v increases indefinitely, and it results, from equation (20), that the temperature T approaches zero. The final temperature of the transformation, T_1, is then zero. Putting $T_1 = 0$ in equation (21), we have, for the total work of the expansion,

$$\theta = EcT_0. \tag{22}$$

The work is therefore equal to the product of the mechanical equivalent of heat, the specific heat under constant volume, and the initial absolute temperature.

We may thus say that the measure of this work is found by multiplying the mechanical equivalent of heat by the quantity of heat which the gas absorbs, under constant volume, when its temperature is raised from the absolute zero to T_0; or gives out, under constant volume, when its temperature is lowered from T_0 to absolute zero.

Examples.

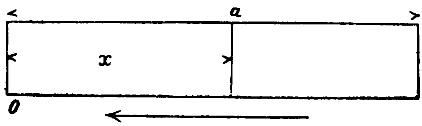
1. Prove that, in adiabatic transformations,

$$\left(\frac{v}{v_0}\right)^{n-1} = \frac{T_0}{T} = \left(\frac{p_0}{p}\right)^{\frac{n-1}{n}}.$$

2. To determine the motion of a gun in recoiling when it is restrained by an air cylinder alone. Since the whole recoil is performed in so short a time as to prevent the conduction outwards of any large amount of heat, the compression of the air may be assumed to be adiabatic. This gives

$$pv^n = p_0 v_0^n = p_0 (Aa)^n; \tag{1}$$

where p_0 is the initial pressure on unit area, A the effective area of the piston, and a the length of the cylinder containing air before recoil.



Let the figure represent the air cylinder, the recoil being in the direction of the arrow. Take the origin of x at O, and take x positive to the right. Then from (1),

$$p = \frac{p_0 a^n}{x^n}.$$

Or, if P be the whole pressure of the air on the piston,

$$P = \frac{p_0 a^n A}{x^n}. \tag{2}$$

Hence,
$$\frac{d^2x}{dt^2} = \frac{p_0 a^n A}{M} \cdot \frac{1}{x^n}, \tag{3}$$

where M is the mass of the gun and carriage. From (3),

$$\int_V^{V_0} \frac{2dx\,.\,d^2x}{dt^2} = \frac{2p_0 a^n A}{M} \int_x^a \frac{dx}{x^n}; \tag{4}$$

where V_0 and V are the initial and any subsequent values of the velocity of recoil. From (4),

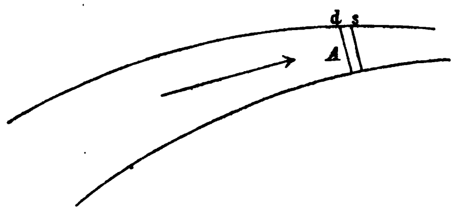
$$V_0^2 - V^2 = \frac{2p_0 a^n A}{M(n-1)}\left[\frac{1}{x^{n-1}} - \frac{1}{a^{n-1}}\right]. \tag{5}$$

By putting $V = 0$, and solving for x, we find the whole length of recoil. The corresponding value of P will be given by (2). Since, before the gun will be run out, a considerable time will elapse and

thus heat will escape, a more reasonable expression for the force tending to return the gun to its position for firing will be found by putting $n = 1$ in (2), and inserting the value of x found from (5) by putting $V = 0$.

In practice, of course, the recoil would never be restrained by an air cylinder alone. If other forces of restraint exist, the above discussion would be modified by properly expressing these forces, and writing these in the second term of (3).

3. To derive an expression giving the velocity of flow of perfect gases. As in the figure, draw any



surface cutting all the stream lines of the gas at right angles, and a second surface parallel to the first and at a distance ds from it. Let $A =$ area of cutting lamina, $ds =$ its thickness, $v =$ the volume of unit weight in it, $V =$ its velocity, $dp =$ the difference of pressures on its two surfaces, and $g =$ the acceleration of gravity. Then $A \cdot ds$ is the volume of the cutting lamina, Adp the moving force applied to it, $\frac{1}{gv}$ the mass of unit weight of gas in the lamina, and $\frac{A \cdot ds}{gv}$ the constant mass of the lamina. We have then

$$\frac{dV}{dt} = \frac{VdV}{ds} = \frac{d}{ds} \cdot \frac{V^2}{2} = \frac{-Adp}{\frac{Ads}{gv}} = -\frac{gvdp}{ds}.$$

4. To find the velocity of escape of gas through the vent of a chamber in which the pressure is 3000 atmospheres, the external pressure being one atmosphere. If the phenomenon is adiabatic, we have $pv^n = p_0 v_0^n$; whence, from the result in the last example, if the velocity of the gas is zero when $p = P$, and is V when $p = p_0$,

$$V^2 = p_0 v_0 \frac{2gn}{n-1}\left[\left(\frac{P}{p_0}\right)^{\frac{n-1}{n}} - 1\right].$$

Here we have, taking metres, kilograms, and seconds as units, $p_0 = 10333$, $v_0 = \frac{1}{1.2932}$, $g = 9.81$; and also take $n = 1.4$, which is

for perfect gases. Then

$$V = 2207 \text{ metres per second, nearly.}$$

5. The area between the curve $pv^n =$ constant, and the axis of v, taken between any finite limit and the limit $v = \infty$, is infinite if $n = 1$, and is finite if $n > 1$. What is the physical interpretation of this?

6. Draw the bore of a gun; and if the charge upon being fired turns instantly into a perfect gas whose pressure is 42 tons per square inch, and then expands adiabatically, trace over it the curve of pressure.

7. Having given that the barometer is 0.7596 metre, and thermometer 18.5° C.; find, from (3), the weight in kilos of a cubic metre of dry air. The weight of a cubic metre of mercury is 13596 kilos, and the weight of a cubic metre of air when the barometer stands at .760 and the thermometer at 0° C. is 1.2932.

Ans. 1.2105 kilos.

CHAPTER III.

THE FORCE OF EXPLOSIVE SUBSTANCES.

23. When a substance is exploded in a capacity of constant volume, the products of combustion develop a pressure whose value, variable with the time, reaches a maximum in an extremely short space of time, and then decreases progressively in consequence of cooling. At the end of this cooling the substance is changed into products whose nature is variable with the conditions of the explosion, and which may be determined by chemical analysis.

These products are, according to circumstances:

Exclusively gaseous (detonation of chloride of nitrogen); gaseous and solid (explosives of the nitrate and chlorate class); gaseous and liquid (nitro-glycerine and gun-cotton).

24. Force of an Explosive Substance.—If we suppose that the laws of Mariotte and Gay-Lussac are applicable to the gasified products of explosion at the instant of maximum tension, the value of this tension, as a function of the temperature and specific volume corresponding, is easily obtained.

Suppose, for example, that we consider unit of weight of an explosive detonating in volume v. The maximum pressure will be given by the equation

$$pv = \frac{p_0 v_0 T_0}{273}, \tag{23}$$

in which T_0 is the absolute temperature of the products of explosion at the instant of maximum tension; v_0 the specific volume of the mixture of these products, supposed entirely gasified.

Consequently, we have the pressure, which is inversely proportional to the volume, equal to $\frac{p_0 v_0 T_0}{273}$, when the volume is reduced to unity.

The quantity

$$f = \frac{p_0 v_0 T_0}{273} \tag{24}$$

is what we shall call the force of the explosive. It represents the

pressure developed by the unit of weight of the substance detonating in unit of volume.

We shall next see how its approximate value may be determined.

25. Heat and Temperature of Combustion.—We shall call heat of combustion the quantity of heat Q that unit of weight of the explosive substance evolves, under constant volume, in the ideal case where the final temperature of the products of combustion is absolute zero.

The absolute temperature of combustion is the temperature T which the products would have if all the heat of the combustion were used to heat them from absolute zero.

If the specific heat of the products of combustion under constant volume were constant throughout the transformations which these products undergo in passing from absolute zero to T, or, inversely, from T to absolute zero, we would have between Q and T the well known relation

$$T = \frac{Q}{c}. \tag{25}$$

The specific heat c would be practically constant if, in the changes undergone, the products remained gaseous, and if, in this state, the hypothesis of Clausius were really applicable to them. In fact, the specific heat of a compound gas being, according to this hypothesis, the same as if the combined elements were mixed, it is clear that the mean specific heat of the mixture preserves, in all states, a value equal to that of the entirety of the simple bodies which make up the explosive body, supposed freely mixed.

In reality, there is reason to believe that this is only approximate, and that the specific heat increases in transformations in which the constituents pass from a less to a greater state of complexity. It also certainly increases at the time of the passage of a body from the gaseous to the liquid state, and from the liquid to the solid state.

Consequently, equation (25) would give too great a value for T, if we take for c a value derived by the hypothesis of Clausius for the entirety of the products supposed gaseous. We would have too small a value, on the other hand, if we follow Bunsen and Schischkoff, and take the value relative to the final state.

We shall adopt the first value, which, as Berthelot has remarked, seems to be preferable for reactions whose temperatures are high, and which, moreover, has the advantage of enabling us to calculate

the value *a priori* from the composition of the explosive substance and independently of an analysis of the products of explosion.

26. We shall suppose also, following all the authors who have treated subjects analogous to the one we have in view, that the temperature, T_0, corresponding to the maximum pressure, is sensibly equal to the temperature of combustion, T, already defined. We write, consequently,

$$T_0 = \frac{Q}{c}. \tag{26}$$

Thus we have a superior limit; for the heat of combustion, Q, comprises not only the quantities of heat which change the temperature of the products, but also those which the chemical or physical changes of state produce.

These principles being admitted, we obtain the definite expression of the force of an explosive substance in this form,

$$f = \frac{1}{273} \frac{p_0 v_0 Q}{c}. \tag{27}$$

The force, then, *is directly proportional to the heat of combustion and the specific volume, and inversely proportional to the specific heat.*

The following are the values of c for the various powders made in France:

Kind of Powder.	Mixture.			Value of c.
	Nitre.	Sulphur.	Charcoal.	
Best sporting powder,	78	10	12	.1452
Cannon powder,	75	12.5	12.5	.1437
Fine-grained powder, called 1 B,	74	10.5	15.5	.1468
Powder of commerce,	72	13	15	.1448
Ordinary blasting powder,	62	20	18	.1420

These values differ very little, and may all be taken as equivalent to their mean, .1445.

It has also been found that for a mixture of 55 parts of picrate of potassium and 45 of saltpetre, $c = .1661$; and, for a mixture of equal weights of picrate and chlorate of potassium, $c = .1513$.

NOTE.—In the calculations relative to gunpowder the charcoal has been treated as pure carbon. The charcoal used in the manufacture of gunpowder contains, however, hydrogen and oxygen about in the proportion in which they exist in water. This will cause an increase

in the value of c, and one which may become appreciable, since the value of c for vapor of water is .4072.

27. Experimental Determination of the Heat of Combustion.—To determine by experiment the heat of combustion of an explosive substance we have only to fire a known weight of it in a vessel plunged in a calorimeter, and to observe the increase of the temperature of the bath when it has absorbed the heat of the reaction.

The first determination of this nature was made by Bunsen and Schischkoff upon a powder like our sporting powder. The experiment showed that the products of combustion of a kilogram of this powder give out 644 *calories* in cooling to a temperature of 20 degrees.

We have attempted, in a series of experiments at the central depot of the State manufactory, to resume these researches, and to extend them to various substances. We shall give, further on, the results of these experiments.

Calorimetric determinations do not give all the heat of combustion, Q. The measured quantity, q, is that which is given out by the products of combustion in passing from the temperature of combustion, T, to the absolute temperature of the calorimeter, t. The difference, $Q - q$, represents, then, the heat which would be given up in passing from t to absolute zero. It is generally very small, compared to Q, because of the extreme elevation of the temperature of combustion. We may, moreover, easily find its value if we know the specific heat, c, between the limits of t and zero. We would have

$$Q = q + ct. \tag{28}$$

The specific heats corresponding to the lower temperatures are, in general, not known; for the want of more precise data, we may substitute for them the theoretical values found in No. 7. These values are too small; but the resulting error, affecting only a relatively very small term, would be small. The mean temperature of our experiments being 17 degrees, the corresponding value of the absolute temperatures is $t = 273 + 17 = 290$.

We may thus, for these elements, approximately correct the observed heats, and derive from them the corresponding heats of combustion.

The following table gives for the principal explosives:

1st. The quantity of heat q, measured by the calorimeter.

2d. The heats of combustion Q, deduced from the foregoing by (28).

3d. The temperatures of combustion calculated from the heats of combustion Q, and the theoretical specific heats by the formula $T = \frac{Q}{c}$.

Name of Explosive.	Heat of Combustion. Measured. q.	Heat of Combustion. Corrected. Q.	Temperature of Combustion.
	Calories.	*Calories.*	*Degrees cent.*
Best sporting powder,	807	849	5870
Cannon powder,	553	795	5500
Fine-grained powder, called B,	731	773	5350
Commercial powder,	694	736	5090
Ordinary blasting powder,	570	612	4240
Chloride of nitrogen,	16	339	4180
Nitro-glycerine,	1720	1784	8120
Gun-cotton,	1056	1123	4850
Picrate of potassium,	787	840	4590
Mixture of 55 picrate of potassium and 45 saltpetre,	916	964	5810
Equal parts of picrate and chlorate of potassium,	1180	1224	8090

These figures are the results of calorimetric experiments made at the central depot, except those for the heat of combustion of chloride of nitrogen, which were made by Deville and Hautefeuille.

28. Approximate Expression for the Force of an Explosive Substance.—We may obtain an expression for the force of an explosive substance. It is only approximate, but is extremely simple, and has the advantage of containing only quantities which may be determined by direct experiment, without its being necessary to have recourse to the determination by chemical analysis; which, as the foregoing considerations show, depend upon the theoretical and somewhat unreliable values of the specific volumes.

Take the general expression for the force,

$$f = \frac{1}{273} \cdot \frac{p_0 v_0 Q}{c}.$$

From equation (14), which is between the heat and specific volume of a gas, we find

$$\frac{1}{273} \frac{p_0 v_0}{c} = E\left(\frac{c'}{c} - 1\right).$$

Consequently, calling n the ratio $\frac{c'}{c}$ of the two specific heats, we have the very simple formula

$$f = (n - 1)\, EQ. \tag{29}$$

The ratio n is equal to 1.40 about for hydrogen, nitrogen, oxygen, carbonic oxide, etc. We have, consequently,

$$f = \frac{2}{5} EQ, \tag{30}$$

if these gases, or others approaching nearly the state of a perfect gas, were the only products of combustion. But, in a great many cases, these products comprise, besides, vapor of water, and other even more complex vapors, which are liquefied or solidified after the cooling. For these compounds n is less than 1.40, and approaches unity, in consequence of the progressive formation of the final state.

If, then, we call ε the weight of permanent gas due to the combustion of a kilogram of the explosive substance, the quantity $n - 1$ may, in a first approximation, be considered as a function of ε, which, vanishing with ε, becomes $\frac{2}{5}$ for $\varepsilon = 1$. We have then, nearly, $n - 1 = \frac{2}{5}\varepsilon$; and, consequently,

$$f = \frac{2}{5} EQ\varepsilon. \tag{31}$$

This formula shows that, similarly to a law of Berthelot, the force of explosive substances is nearly proportional to the product of the heat of combustion by the weight of the permanent gases produced by the explosion.

If the conditions of the explosion are such that the substance is entirely, or even partially, dissociated, so as to produce compound gases for which we would have $n = 1.40$, as would be the case for carbonic oxide, we must take $\varepsilon = 1$, and the force would be given by (30).

29. By means of experiments made at the central depot, ε has been determined for some substances. These, together with the heats of combustion, and the corresponding forces, expressed in atmospheres, are tabulated below:

Name of Explosive.	Weight of Gas. ε.	Heat of Combustion. Q.	Force. f.
	Kil.	*Cal.*	*Atm.*
Sporting powder,	.337	849	4.83
Common powder,	.412	795	5.53
Fine-grained powder, called B,	.414	773	5.40
Powder of commerce,	.446	736	5.54
Ordinary blasting powder,	.499	612	5.15
Chloride of nitrogen,	1.000	339	5.72
Nitro-glycerine,	.800	1784	24.09
Gun-cotton,	.853	1123	16.16
Picrate of potassium,	.740	840	10.49
55 parts of picrate of potassium and 45 of saltpetre,	.485	964	7.89
Equal parts of picrate and chlorate of potassium,	.466	1224	9.63

30. Relative Force of the Explosive Substances.—It is remarkable that the five powders should have about the same force, notwithstanding differences of fabrication. This result has been confirmed by experiment, which shows that the bursting charges of the five different powders for the same shell vary only 15 to 17 grams.

The mean of the forces of powders is 5.29. This corresponds to a force of 5290 atmospheres for a kilogram of powder, detonating in a litre, that is to say, in its own volume; these results are thus in accord with those made by Captain Noble upon powder exploded in its own volume. The pressure measured by him under these conditions by means of a gauge was 37 tons on the square inch, or 5600 atmospheres, for the F. G. powder, and about 32 tons for the R. L. G. The following table shows the relative force of the different explosives:

Name of Explosive.	Relative Force.
Powder containing saltpetre,	1.
Chloride of nitrogen,	1.08
Nitro-glycerine,	4.55
Gun-cotton,	3.06
Picrate of potassium,	1.98
55 parts picrate of potassium and 45 of saltpetre,	1.49
Equal parts picrate and chlorate of potassium,	1.82

31. The figures of this table appear to agree sufficiently well with the real effects of explosives, fired in a manner to produce what, in the experiments undertaken with M. Roux, we have called explosions *of the second order*, that is to say, explosions produced by any other agency than a strong fulminating primer.

For example, the relative force of gun-cotton is about equal to that found by the French commission on gun-cotton (3.20), derived from the comparison of the charges just necessary to burst a shell.

Other similar experiments confirm the result for nitro-glycerine. The mean charge of powder required to rupture shell was 16 grams. In the same shells, the effect of dynamite which contained 50 per cent of nitro-glycerine, fired by a small quantity of powder, was such that 1 part of nitro-glycerine was equal to 2 parts of powder, under these circumstances. But, as Berthelot has remarked, the heat disengaged by firing dynamite is divided between the products of the

explosion of the nitro-glycerine and the inert vehicle, which latter has about the same specific heat. It results that, in such dynamite, the temperature, and consequently the force, must be lowered one-half. The force of pure nitro-glycerine should then be doubled, and be estimated at about 4. The difference between this value and that given in the table is not great. We will add that, for a mixture of picrate and nitrate of potassium, the bursting charge was found to be about 11 grams, which corresponds to a relative force of $\frac{16}{11} = 1.45$; which differs little from the theoretical value, 1.49.

The table of paragraph 29 may be used to furnish examples of the theoretical computation of f, the force of the powder.

CHAPTER IV.

THE PRESSURE PRODUCED BY THE POWDER GASES IN A VESSEL WHOSE CAPACITY REMAINS CONSTANT.

32. MM. Noble and Abel communicated (1876) to the Academy of Sciences, the summary of important researches on the combustion of powder. Admitting the results of these learned experimenters, that which relates to the pressures developed by the combustion of the powder in a closed vessel is particularly remarkable. It results, in fact, from the measure of the pressures that we are able to explain the facts observed, by supposing:

1st. That a part of the products of combustion is in a solid state.

2d. That the pressure due to permanent gases only can be calculated after the law of Mariotte, by deducting from the volume of the envelope that of the solid residue.

33. The calculation of the pressures, under this hypothesis, is done in the following manner: Let

$C =$ the volume of the vessel or place of combustion.

$\varpi =$ the weight of the powder burned.

$T_0 =$ the absolute temperature of the products of combustion.

$a =$ the volume at this temperature, of the solid residue furnished by the combustion of one kilogram of powder.

$v_0 =$ the volume (at zero centigrade, and under the normal atmospheric pressure) of the permanent gases of a kilogram of powder.

The permanent gases produced by the charge ϖ of powder at T_0 occupies a volume equal to $C - a\varpi$. In order to find the corresponding pressure, it is necessary to apply the laws of Mariotte and Gay-Lussac, observing that the same gases under the normal pressure p_0 and at 273° of absolute temperature occupy a volume equal to ϖv_0; we have then

$$\frac{p}{p_0} = \frac{\varpi v_0}{C - a\varpi} \cdot \frac{T_0}{273},$$

or better,

$$p = f \frac{\varpi}{C - a\varpi}, \qquad (31)$$

putting for brevity,

$$f = \frac{p_0 v_0 T_0}{273}. \tag{32}$$

We can also introduce the *density of loading*, that is to say, the ratio $\Delta = \frac{\varpi}{C}$ of the weight of the powder to the capacity of envelope, and write

$$p = \frac{f\Delta}{1 - a\Delta}. \tag{33}$$

34. Force of the Powder.—The quantity f defined by the relation (32) represents the pressure of the permanent gases of a kilogram of powder, occupying, at the temperature of the flame, a volume equal to unity, which exacts the condition that in taking account of the volume of the solid products, the volume of the envelope should be $1 + a$.

We will call this quantity the *force of the powder.*

In a preceding work, supposing with several other authors that the products of combustion of powder, and more generally of any explosive substance, were totally converted into gases, at the temperature of the explosion, we have called the force of the explosive substance the pressure developed by the unit of weight of the substance burning in the unit of volume.

This definition accords with the preceding, since, under this hypothesis, the volume a of the solid residue becomes zero; but it will be seen that the new definition is more general.

35. The results of the English experiments accord, in fact, very well with formula (33). MM. Noble and Abel have verified this with the pressures which they obtained by burning pebble powder in a closed vessel, the densities Δ varying by increments of 0.1 from $\Delta = 0.1$ to $\Delta = 1.0$.

The summary of their memoir does not mention the values admitted for the values of the constants f and a. These values seem to have been calculated by the aid of the pressures which correspond to the values $\Delta = 0.6$ and $\Delta = 1$.

We have calculated these quantities by using in their determination all the results observed, using French units. We have found:

$$\begin{aligned} a &= 0.6833, \\ f &= 219300, \end{aligned} \tag{34}$$

the units being the *decimetre and the kilogram.*

The following table shows the pressures measured, the pressures calculated by formula (33), and the corresponding differences.

We have divided the pressures by 100, consequently in the table the pressures are in *kilograms per square centimetre.*

Densities. Δ	Pressures. Measured.	Pressures. Calculated.	Differences.
0.1	231	235	− 4
0.2	513	508	+ 5
0.3	839	828	+ 11
0.4	1173	1207	− 34
0.5	1684	1666	+ 18
0.6	2266	2230	+ 36
0.7	3006	2943	+ 63
0.8	3942	3869	+ 73
0.9	5112	5127	− 15
1.00	6567	6926	−359

The differences are not important, and since they do not seem to follow any law, they may be imputed to errors of experiment; besides, the account before mentioned does not give any detail of the method adopted for the measure of the pressures, and it is impossible to discuss the results with exactness. It is likely that the experiments on the pressure of the gases of the powder in a closed vessel are those that were described by Captain Noble in a lecture before the Royal Society of Great Britain, or, at least, that they were determined in the same manner, that is, by *crusher gauges*, and the readings of these instruments being open to discussion, the results found should be accepted with some reserve. We add that the results differ completely from those which Rumford has deduced from his experiments.

In the meantime, formula (33) represents the facts with an exactness such that it is allowable to consider as very plausible the hypothesis according to which the products of the combustion of the powder will be, *under certain circumstances*, partly solid and partly gaseous. The pressure observed being due to the permanent gases alone, may be calculated by Mariotte's law, taking into account the volume occupied by the solid residue.

We show in the following chapter, how, under this hypothesis, the equation of the movement of the projectile in the interior of a gun may be established.

36. Temperature of the Products of Combustion.—The value of f being determined by experiment, we deduce from it by equation (32) the value of T_0. We have, in fact,

$$T_0 = \frac{273 \cdot f}{p_0 v_0}.$$

According to MM. Noble and Abel, the value of v_0 was 280 litres for the powder experimented with. By making

$$f = 219300,$$
$$v_0 = 280,$$
$$p_0 = 103.33,$$

we find for the absolute temperature of the gases,

$$T_0 = 2070°.$$

37. In order to calculate this temperature theoretically, it is necessary to know:

1st. The heat lost by the products of the combustion of the unit of weight of powder without production of exterior work by being lowered from the temperature T_0 of combustion to a determined temperature, say zero centigrade, or 273° of absolute temperature.

2d. The mean specific heat of the products of combustion between these limits of temperature.

In fact, designating by Q the heat of combustion, and by c the specific heat, we have the relation

$$Q = c(T_0 - 273),$$

whence

$$T_0 = 273 + \frac{Q}{c}.$$

The heat of combustion can easily be determined by the burning of powder in a calorimeter. MM. Noble and Abel found it equal to 705 *calories* (French units of heat) for the experimental powder.

With regard to the specific heat, it is unknown. MM. Bunsen and Schischkoff have admitted the value $c = 0.185$ for the products of combustion of a powder similar to our sporting powder. But this quantity corresponds to a temperature nearly the mean atmospheric temperature, and, following MM. Noble and Abel, it should vary—*increase*, probably—with the temperature.

Therefore, in default of more precise data, we admit the value $c = 0.185$ with $Q = 705$, and we find

$$T_0 = 4080°,$$

that is to say, a value nearly double that which has been deduced

from the measure of the pressures. One would be led to admit, then, that the mean specific heat has a value nearly double that which MM. Bunsen and Schischkoff have adopted, and, consequently, since the specific heat of the gases under constant volume is independent of the temperature, that the specific heat of the solid residue at the temperature of combustion is more than double what it is at ordinary temperatures.

The value of c used must then be too small if we admit the value $T = 2070°$, and it should then be 0.307. This value, in fact, is that which should be used; and on the hypothesis that c remains the same for various powders, the formula $T_0 = 273 + \frac{Q}{c}$ enables us to compute the temperatures of combustion of various powders as well as their *force.*

The following table gives the temperature of combustion of various powders:

Kind of Powder.	Specific Volume of the Gas.	Heat of Combustion.	Temperature of Combustion.
Pebble Powder, R. L. G.	269 dc.	705 c.	2570
Sporting Powder, . . .	234	810	2911
Cannon Powder, . . .	261	756	2735
C. and S. P. Powder, . .	280	732	2657
Mining Powder, ...	316	574	2142

Examples.

1. The VIII-inch B. L. R. has a chamber capacity of 3824 cubic inches; hence, when loaded with 125 lbs. of powder, $\Delta = .905$. Having given that when so loaded (cocoa powder, and projectile weighing 250 lbs.), the measured maximum pressure was 31,500 lbs. per square inch; find, from (33), where the shot's base must have been at the instant of maximum pressure. We have

$$\Delta = \frac{\varpi}{\omega u_0 + \omega u} = \frac{p}{f + ap},$$

hence

$$u = \frac{\varpi}{\omega}\left(\frac{f}{p} + a - \frac{\omega u_0}{\varpi}\right).$$

Ans., the shot had travelled 39.2 inches.

2. The volume of the bore of the above gun being 13,732 cubic inches, find the pressure just as the shot clears the muzzle.

At this time, $\Delta = .252$; and pressure $= 9495$ lbs. per square inch.

3. Trace the locus of (33), and explain why it has an asymptote when $\Delta = \frac{1}{a}$.

4. Draw the bore of a gun, and, from (33), trace a curve of pressures over it.

5. If the safe internal pressure in a gun be given by

$$P = \frac{3\theta(R_1^2 - R_0^2)}{4R_1^2 + R_0^2},$$

the gun consisting of one tube only; derive from (33) the correct profile of the outside of the gun.

To find the powder charge necessary to rupture the walls of shells, the necessary pressure is found by the formula in Example 5, where θ = tensile strength of metal, and R_1 and R_0 the external and internal radii at the weakest point. From Example 1 we have $\Delta = \frac{P}{f + aP}$, in which the value of f for shell powder may be taken at 37 tons per square inch (Noble's experiments with F. G. powder), and $a = .6$; hence, if S is the capacity of the cavity in the shell in cubic inches, we have for ϖ in pounds,

$$\varpi = .036132 \frac{SP}{37 + .6P}.$$

If any other explosive is used, the relative *force* may be found from the table in paragraph 30. Dynamites contain a certain percentage of nitro-glycerine per unit weight, which percentage is always known in practice; therefore the weight of dynamite to perform certain work in exploding shells may thus be obtained.

6. The tensile strength of forged steel being assumed to be 100,000 pounds per square inch, what charge of powder is necessary to just rupture the walls of the 6-inch B. L. R. steel shell; the capacity of the cavity being assumed to be 50 cubic inches, and the internal radius of the cavity 1.5 inches? *Ans.* .8359 lb.

7. What weight of 50 per cent dynamite will rupture the walls of the shell in Example 6? *Ans.* .23 lb. nitro-glycerine.
.46 lb. 50 per cent dynamite.

8. Suppose the strength of the walls of a 6-inch gun to be 30 tons per square inch, how much gun-cotton confined in a space of 55 cubic inches in the bore will just suffice to destroy the gun?
Ans. .46 lb.

To find the amount of work which the permanent gases generated by the explosion of powder are capable of doing in a gun, assuming that the gases expand without loss of heat to the walls of the gun.

The fundamental equation for permanent gases being

$$pv = RT, \qquad (a)$$

where p, v, and T are respectively the pressure, volume, and temperature reckoned from absolute zero, and R is a constant.

In (7) and (8),

$$RdT = pdv + vdp; \qquad (b)$$

$$dq = cdT + \frac{c' - c}{R} pdv. \qquad (c)$$

Let c_1 be the specific heat of the solid residue of the exploded powder, and β the ratio of its weight to that of the gaseous products of the explosion; it is clear that the solid products will lose an amount of heat $\beta c_1 dT$ while the temperature changes dT. Now, by the hypothesis of Noble and Abel, the heat given up by the solid residue is absorbed by the gases; therefore

$$dq = -\beta c_1 dT.$$

Equation (4) then becomes

$$-(c + \beta c_1) dT = \frac{c' - c}{R} pdv.$$

Substituting the value of dT from (3), we find, after reduction,

$$-(\beta c_1 + c') pdv = (\beta c_1 + c) vdp; \qquad (d)$$

whence, separating the variables,

$$-(\beta c_1 + c') \frac{dv}{v} = (\beta c_1 + c) \frac{dp}{p}. \qquad (e)$$

Since c', c, c_1, and β are assumed to be constant throughout the range of pressure and volume considered, we have

$$-(\beta c_1 + c') \int_{v_0}^{v} \frac{dv}{v} = (\beta c_1 + c) \int_{p_0}^{p} \frac{dp}{p}, \qquad (f)$$

where the limits of integration are the initial and any subsequent values of v and p.

From (7),

$$p = p_0 \left[\frac{v_0}{v} \right]^{\frac{\beta c_1 + c'}{\beta c_1 + c}} \qquad (g)$$

which gives the law connecting the pressure and volume.

For the work, we have

$$W = \int_{v_0}^{v} pdv,$$

whence
$$W = p_0 v_0^{\frac{\beta c_1 + c'}{\beta c_1 + c}} \int_{v_0}^{v} \frac{dv}{v^{\frac{\beta c_1 + c'}{\beta c_1 + c}}}, \tag{h}$$

and, integrating,
$$W = \frac{p_0 v_0 (\beta c_1 + c)}{c' - c} \left[1 - \left(\frac{v_0}{v} \right)^{\frac{c' - c}{\beta c_1 + c}} \right] \tag{i}$$

In this expression, v_0 and v must be corrected, as they should represent, not the whole volume of the powder chamber, but the fraction of it occupied by the permanent gases given off. Let a be the ratio between the volume of the solid products of the explosion and that of the powder before explosion, then (10) becomes,

$$W = \frac{p_0 v_0 (1 - a)(\beta c_1 + c)}{c' - c} \left[1 - \left(\frac{v_0 (1 - a)}{v - a v_0} \right)^{\frac{c' - c}{\beta c_1 + c}} \right]. \tag{k}$$

The following are the values of the constants which it contains, in grams and centimetres:

$$p_0 = 6532450 \qquad c' = .2353$$
$$v_0 = 1 \qquad \beta = 1.3148$$
$$a = .6 \qquad c_1 = .4090$$
$$c = .1782$$

Equation (11) may then be written

$$W = 32763000 \left[1 - \left(\frac{.4}{v - .6} \right)^{.07976} \right].$$

The result is evidently in centimetre-grams per gram of powder burned. To convert this to foot-tons per pound burned, multiply by $\frac{1}{2240 \times 30.48}$.

CHAPTER V.

THE COMBUSTION OF A CHARGE OF POWDER. THE LAW OF BURNING OF GRAINS OF VARIED FORM IN FREE AIR.

38. In the case of a prism of powder burning at one end only, the combustion takes place in successive layers, and with a velocity which depends upon the nature of the powder, on the density, and upon the pressure under which the combustion takes place.

39. Let us now consider the case of a charge of powder composed of a number of powder grains.

Before entering upon a mathematical discussion of this subject it is necessary to assume that the combustion takes place in free air, that the grains are all of the same density and homogeneous throughout all their mass, and that the flame is propagated instantly upon the whole surface of the grains, also that the combustion then takes place in successive layers in each of the grains.

The *velocity of emission* may be defined as the ratio of the weight of powder burned in a small increment of time to the time itself, *i. e.* if at a given instant V is the velocity of combustion of the matter and S the total surface in ignition, the volume of powder burned in a small increment of time dt, if the density of the powder is ρ, is represented by $V\rho S dt$ ∴ the *velocity of emission*, or the ratio

$$\frac{V\rho S dt}{dt} = V\rho S.$$

The *velocity of emission* at any instant depends—

1st, on the velocity of combustion at that instant;

2d, on the density of the material of the grain;

3d, on the total remaining surface in ignition; and if we suppose the density constant, and that the combustion takes place in free air, we may suppose V also constant, and the *velocity of emission* will depend absolutely on the total remaining surface in ignition; it is on this hypothesis, therefore, that we seek for the law of the combustion of a charge of gunpowder.

40. Spherical Grains.—First, take the case of spherical grains, all of the same size, and what can be proved with regard to spherical grains will also hold approximately true for grains of the form of any regular polyedron. Let R be the radius of the sphere (or of the inscribed sphere in the case of grains of the form of a regular polyedron). Let R be divided into, say, 10 equal parts; then at the end of successive equal periods of time we have by hypothesis that R will be reduced to .9, .8, .7, .6, .5, .4, .3, .2, .1 of R; but the remaining surfaces in ignition will be represented by the squares of these numbers if the initial surface was 1, or .81, .64, .49, .36, .25, .16, .09, .04, .01, and if the grains of a charge are ignited simultaneously it follows also that the successive *velocities of emission* are proportional to these numbers.

The rapid decrease of the *surface* of emission, and consequently the velocity of emission, in the case of spherical grains, or of a regular polyedral form, at the beginning of combustion is worthy of notice.

The volume of powder burned as a function of the time may be calculated as follows: The initial volume being $\frac{4}{3}\pi R^3$, at the end of the time t it is reduced to $\frac{4}{3}\pi(R - Vt)^3$, V being the velocity of combustion of the material of the grain in free air.

The volume burned at the end of t is

$$\frac{4}{3}\pi R^3 - \frac{4}{3}\pi(R - Vt)^3,$$

or

$$\frac{4}{3}\pi R^3 - \frac{4}{3}\pi\left(1 - \frac{V}{R}t\right)^3.$$

Now, if τ is the time for the total combustion of the grain, $R = V\tau$ and $\frac{V}{R} = \frac{1}{\tau}$ $\therefore$ the expression for the volume burned becomes

$$\frac{4}{3}\pi R^3\left(1 - (1 - \frac{t}{\tau})^3\right);$$

the ratio of the volume burned to the initial volume is then

$$1 - \left(1 - \frac{t}{\tau}\right)^3,$$

and if we suppose that instead of a single grain, a charge is composed of grains ignited simultaneously, the weight of the material burned in one pound of the charge as a function of the time may be represented by this same expression.

∴ The curve (Fig. 1) $y = 1 - \left(1 - \frac{t}{\tau}\right)^3$ represents the law of combustion of a charge of spherical grains or of grains of a regular polyedral form.

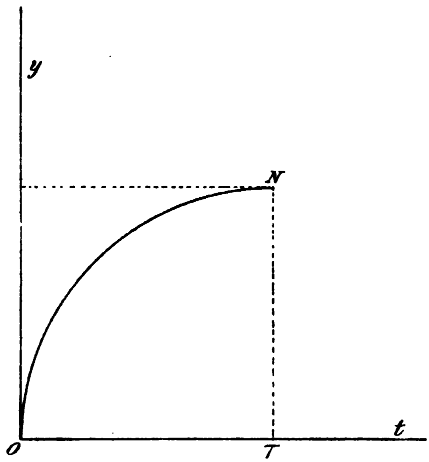


FIG. 1.

For a charge of a given weight W, the weight of the charge burned as function of the time of burning is

$$W\left\{1 - \left(1 - \frac{t}{\tau}\right)^3\right\}.$$

If the charge is composed of irregular grains, instead of those of regular form equal to each other, it may be admissible in practice, for the purpose of the study of combustion, to suppose the same number and same weight of spherical grains.

If the size of the grain is increased, the initial surface of a charge of grains is *decreased* ∴ the rate of decrease of the velocity of emission in free air is diminished. Take a charge composed of cubical grains, conceive that each grain is divided into groups of 8 small cubes, each of one-half the dimensions of the original cube. The weight of the charge remains the same, but the initial surface in ignition in the large cube is only one-half as great and the period of combustion is twice as great as in the case of the small cubes, since the thickness is twice as great. The decrease of surface, and consequently the velocity of emission, is therefore less rapid in the large cube than for grains one-eighth its size. This may be shown to be true of grains of whatever form, since in similar solids, if their linear dimensions are

to each other as $1 : n$, their surfaces are to each other as $1 : n^2$, and their volumes as $1 : n^3$.

This curve passes through the origin, for if $t = 0$, $y = 0$, and rises rapidly as the value of y increases rapidly for small values of t. When $t = \tau$, it is tangent to a line parallel to the axis of t.

The *derivative* of y with respect to t is only another expression for the *velocity of emission*, since it is the limit of the ratio of the weight of gas emitted to the corresponding interval of time; and since the latter is supposed infinitely small, we have

$$z = \frac{3}{\tau}\left(1 - \frac{t}{\tau}\right)^2,$$

and the curve represented by this equation is as shown in Fig. 2.

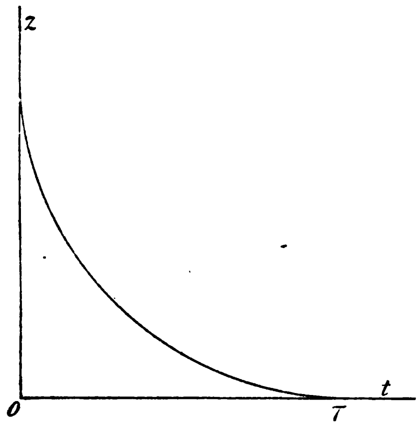


FIG. 2.

When $\qquad t = 0, z = \frac{3}{\tau},$

when $\qquad t = \tau, z = 0.$

41. Parallelopiped.—To find the law of combustion in free air of a grain of the form of a parallelopipedon, if α, β, μ are the dimensions of the grain, its primitive volume is $\alpha\beta\mu$ at the end of a time t, the dimensions are each diminished by $2Vt$, and the remaining volume is then

$$(\alpha - 2Vt)(\beta - 2Vt)(\mu - 2Vt),$$

and the volume burned is

$$\alpha\beta\mu - (\alpha - 2Vt)(\beta - 2Vt)(\mu - 2Vt),$$

or

$$\alpha\beta\mu - \alpha\beta\mu\left(1 - \frac{2Vt}{\alpha}\right)\left(1 - \frac{2Vt}{\beta}\right)\left(1 - \frac{2Vt}{\mu}\right) \qquad (1)$$

If τ is the time of combustion of a grain in free air, and if α is the smallest dimension of the grain, we have

$$\frac{a}{2} = V\tau, \quad \text{or} \quad \frac{2V}{a} = \frac{1}{\tau},$$

and (1) may be written

$$a\beta\mu - a\beta\mu\left(1 - \frac{t}{\tau}\right)\left(1 - \frac{at}{\beta\tau}\right)\left(1 - \frac{at}{\mu\tau}\right);$$

and if x, y are the ratios of the smallest dimension to the two others, we have

$$a\beta\mu\left\{1 - \left(1 - \frac{t}{\tau}\right)\left(1 - x\frac{t}{\tau}\right)\left(1 - y\frac{t}{\tau}\right)\right\},$$

and the ratio of the volume burned to the original volume is then

$$1 - \left(1 - \frac{t}{\tau}\right)\left(1 - x\frac{t}{\tau}\right)\left(1 - y\frac{t}{\tau}\right),$$

and if we consider a charge composed of grains ignited simultaneously, the weight of powder burned per lb. of charge as a function of the time is this expression.

Ordinarily the grains are of a form having two opposite faces square $\therefore x = y$, and the expression becomes

$$1 - \left(1 - \frac{t}{\tau}\right)\left(1 - x\frac{t}{\tau}\right)^2.$$

The derivative of this expression with respect to t gives

$$z = \frac{1}{\tau}\left(1 - x\frac{t}{\tau}\right)^2 + \frac{2x}{\tau}\left(1 - x\frac{t}{\tau}\right)\left(1 - \frac{t}{\tau}\right),$$

the curve is then CD (in Fig. 3).

When $t = 0$, $\quad z_0 = \dfrac{1 + 2x}{\tau},$

when $t = \tau$, $\quad z_\tau = \dfrac{(1 - x)^2}{\tau} \quad \therefore z_0 - z_\tau = \dfrac{4x - x^2}{\tau}.$

The velocity of emission decreases constantly from z_0 to z, because the second derivative is negative. The difference $z_0 - z_\tau$ may perhaps be taken as a measure of the progressive nature of the powder. This difference diminishes with x and approaches 0 when x tends towards 0. We have by hypothesis that $x < 1$. It is also very evident that in practice it is not possible to decrease the value of x below a certain limit without making the grain too fragile.

If $x = 1$, we have the case of a cubical grain, or

$$z = \left(\frac{1 + 2x}{\tau}\right)\left(1 - \frac{t}{\tau}\right)^2,$$

and the curve is then AB. From which it appears that a flat grain is more progressive than a cubical grain.

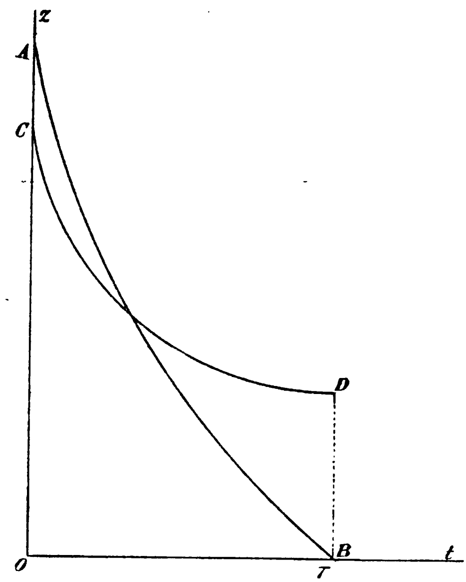


FIG. 3.

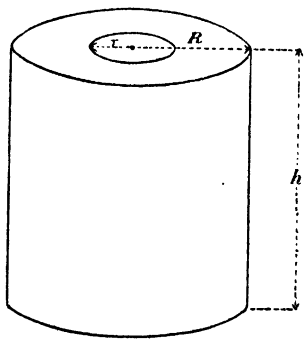


FIG. 4.

42. Pierced Cylinder.—The next case is that of a cylindrical form pierced by a central canal (Fig. 4). It will be seen that the *surface* of emission is decreased for the external part and increased for the internal part, and it is conceived that the *decrease of velocity of emission* will be more diminished in a grain pierced by a canal than in a solid grain.

If R and r are the external and internal radii, the primitive volume is

$$\pi (R^2 - r^2) h.$$

At the end of t, R becomes $R - Vt$,
r " $r + Vt$,
h " $h - 2Vt$,

$\therefore$ the volume at the end of t is reduced to

$$\pi [(R - Vt)^2 - (r + Vt)^2] (h - 2Vt).$$

The volume burned is

$$\pi (R^2 - r^2) h - \pi \{(R - Vt)^2 - (r + Vt)^2\} (h - 2Vt),$$

or $\pi (R^2 - r^2) h - \pi \{R^2 - r^2 - 2(R + r) Vt\} (h - 2Vt)$,

or $\pi (R^2 - r^2) h - \pi (R^2 - r^2) h \left(1 - \dfrac{2Vt}{R - r}\right)\left(1 - 2\dfrac{Vt}{h}\right)$. (1)

If we suppose $R - r$, the annular thickness, is the smaller dimension, and if we designate by τ the whole period of combustion of a grain, we have

$$\frac{R - r}{2} = V\tau \text{ or } \frac{2V}{R - r} = \frac{1}{\tau},$$

then
$$\frac{2V}{h} = \frac{1}{\tau}\left(\frac{R - r}{h}\right). \quad \frac{R - r}{h} = x.$$

Then (1) becomes

$$\pi (R^2 - r^2) h - \pi (R^2 - r^2) h \left(1 - \frac{t}{\tau}\right)\left(1 - x\frac{t}{\tau}\right),$$

and the ratio of the volume burned to the primitive volume is

$$1 - \left(1 - \frac{t}{\tau}\right)\left(1 - x\frac{t}{\tau}\right),$$

and as in other cases before considered, the weight of a lb. of powder burned as a function of the time is given by this same expression.

If the smaller dimension of the grain is the height of the cylinder, the same expression would be true, *provided* we continue to designate by x the ratio of the smaller dimension to the greater.

The derivative with respect to t gives the following:

$$z = \frac{1}{\tau}\left(1 - x\frac{t}{\tau}\right) + \frac{x}{\tau}\left(1 - \frac{t}{\tau}\right).$$

When $\quad t = 0, \quad z_0 = \dfrac{1 + x}{\tau},$

when $\quad t = \tau, \quad z_\tau = \dfrac{1 - x}{\tau} \quad \therefore \quad z_0 - z_\tau = \dfrac{2x}{\tau},$

∴ the difference of velocities of emission varies with x and tends toward 0 as x approaches 0 in value. Having given a minimum dimension which fixes the value of τ, x being < 1, we easily see that for the same value of x, the difference of the *velocities of emission* at the beginning and end of combustion is smaller with a cylindrical grain pierced than for a flat grain of the form of a parallelopipedon, and if this difference measures the progressive nature of the powder, the former is more progressive than the latter, other conditions being equal; but that this conclusion may be true, it is also necessary to suppose that the minimum value of x in practice is the same in both forms, a condition little probable, and in fact the pierced cylindrical form makes the grains fragile.

If it is wished to obtain a definite quantity of work, that is, to obtain a given velocity at the muzzle of a gun for a projectile, it would be found under the best possible conditions if the maximum pressure is comparatively low, and the mean pressure such that the area of the pressure curve remains the same. We must use a powder so constituted that in combustion in free air the emission of gas at the beginning will be small and afterward *slowly* decreasing, and by increasing the weight of the charge, realize the velocity given with a comparatively low pressure on the gun.

On this principle are founded researches concerning *progressive powders*, *i. e.* having a progressive mode of action.

Up to this point it has been considered that the grains are homogeneous throughout their mass, and it has also been assumed that the grains burn with a uniform velocity of combustion in free air, in successive layers from their surfaces to their centres. We have seen that it is possible to obtain progressive properties to a limited extent, by variation of the form of the grain, provided that form is such as will not too easily break up by the pressure of the gases.

43. The Combustion of a Whole Charge.—If we suppose the ignition to be instantaneous throughout the charge, the function which represents the law of burning of a single grain will also represent the law of burning of the whole charge, provided the conditions are those we have considered in the case of single grains. Then if $\psi(t)$ is the fraction of a grain burned after the time t (see equations for values of y), and ϖ is the weight of the charge, if y represents the fraction of the charge burned after the time t, we have the following equation,

$$y = \varpi\psi(t),$$

in which $\psi(t)$ takes the three forms below, according as the charge is composed of grains in the form of spheres, parallelopipeds, or pierced cylinders, respectively:

$$\left.\begin{aligned}
\psi(t) &= 1-\left(1-\frac{t}{\tau}\right)^3, \text{ spheres,}\\
\psi(t) &= 1-\left(1-\frac{t}{\tau}\right)\left(1-x\frac{t}{\tau}\right)\left(1-y\frac{t}{\tau}\right), \text{ parallelopipeds,}\\
\psi(t) &= 1-\left(1-\frac{t}{\tau}\right)\left(1-x\frac{t}{\tau}\right), \text{ pierced cylinders,}
\end{aligned}\right\} \quad (35)$$

in which x and y represent the ratio of the thickness to the other two dimensions in case of parallelopipeds, and x represents the ratio of $R-r$ and h, the smaller to the greater, in the case of pierced cylinders.

44. The formulas for $\psi(t)$ may evidently be put in the following form,

$$\psi(t) = \frac{at}{\tau}\left(1-\lambda\frac{t}{\tau}+\mu\frac{t^2}{\tau^2}+\ldots\right),$$

in which the second term is negative.

Comparing the expansion of $\psi(t)$ with each of the above equations, we have:

$$\left.\begin{aligned}
&\text{For spheres,} && a=3; && \lambda=1; && \mu=\frac{1}{3}\\
&\text{For parallelopipeds,} && a=1+x+y; && \lambda=\frac{x+y+xy}{1+x+y}; && \mu=\frac{xy}{1+x+y}\\
&\text{For pierced cylinders,} && a=+x; && \lambda=\frac{x}{1+x}; && \mu=0
\end{aligned}\right\} \quad (36)$$

The quantities τ, a, λ, μ characterize the mode of combustion of a charge of powder in free air for each kind of powder which may be assimilated to spheres, parallelopipeds, or pierced cylinders. Irregular shaped grains, cubical powder, waffle powder assimilate to spheres; solid prismatic and solid cylindrical powder to parallelopipeds; pierced prismatic assimilates to a pierced cylinder so far as its law of burning is concerned.

a, λ, μ depend alone on the *form* of the grain, but τ depends upon the nature of the powder, the size of the grains, the density, and in general upon all those characteristics which influence the time of combustion.

We shall see later how to take account of the variable pressure under which the grain burns, as is always the case in a gun.

Examples.

1. Find the law of burning of a cylinder of powder of radius R', if it is perforated axially by a cylinder of radius R, and burns in this axial cylinder only.

$$y = 2\frac{R}{R'+R}\frac{t}{\tau} + \frac{R'-R}{R'+R}\frac{t^2}{\tau^2}.$$

2. Show that, as the axial cylinder in Example 1 diminishes, the law of burning there deduced approaches that which must hold in order that the pressure on the shot's base shall be constant.

In case the pressure is constant, the volume containing powder increases with the square of the time; also, if the pressure be constant, the amount of powder burned must vary directly as the volume in which it is contained. Therefore the amount of powder burned must vary as the square of the time.

3. It is usual to assume that the density of grains of gunpowder is uniform throughout their mass. If, however, we assume that it varies along the radius, following the law $\rho = \left(\frac{r}{R}\right)^m \rho_1$ (considering the grain a sphere, for simplicity); where ρ is the density at any point of radius r, R the outside radius, and ρ_1 the density at the outside, we may find the value of m.

To find the mean value of any function, we have, see Johnson's Integral Calculus, art. 97,

$$\int_a^b f(x)\,dx = M(b-a).$$

In the case considered we have

$$M = \frac{\rho_1}{m+1}.$$

If now we take the density at the outside equal to that of a solid homogeneous mass consisting, in the ordinary proportions, of gunpowder, we have $\rho_1 = 1.985$. The mean density is the quantity measured by the densimeter. If we take this 1.6, we find $m = .24$.

4. The quantity y is the *volume* burned. The rate of evolution of *weight* of gas is given by $\epsilon = V\rho S$, as already shown.

Hence, if the grain is a sphere, and $V = \frac{c}{\rho^n}$, and $\rho = \left(\frac{r}{R}\right)^m \rho_1$,

$$\epsilon = 4\pi \cdot c\,\frac{R^{m(n-1)}}{\rho_1^{n-1}} \cdot r^{2-m(n-1)}.$$

Hence, unless $m(n-1) > 2$, ϵ decreases as r decreases.

5. Sketch the bore of a gun, and draw over it lines of pressure and velocity referred (1) to space described and (2) to time elapsed, in case the powder pressure is constant. Draw also lines giving weight of powder burned referred to space and time, in the same case.

CHAPTER VI.

THE GENERAL EQUATION OF MOTION IN THE BORE.

45. If we admit that a part of the products of combustion is in a solid state in the ordinary conditions of the employment of powder in guns, we can, in evaluating the work of the expansion of the gas, make two hypotheses which probably will include the truth:

First, that the solid products are always in equilibrium of temperature with the gas, and thus give up some heat which is converted into work.

Second, that the temperature of the solid products is constant during the expansion of the gas.

The first is that of Noble and Abel, and the second that of Bunsen and Schischkoff, and neither one is strictly true, probably.

Sarrau adopts the second as being nearer the truth than the first; but, as we shall see, an exact determination of the truth or falsity of the assumption is not necessary, since the constants which enter into the equations for the computation of velocities and pressures may be modified to correct any error made in the assumption. For a full discussion of this question see Proceedings U. S. Naval Institute, No. 28, pp. 87 to 99.

46. Suppose now the inflammation of the charge to be instantaneous, that is to say, that the flame is instantly propagated throughout the whole charge. We may make this assumption in comparing the time of ignition of the whole charge with that of the combustion of the grains, especially in the case of large-grained powder.

Let y denote the weight of powder burned in the time t, counting from the instant the charge is ignited.

T_0, the temperature of the gases when they are formed.

T, the temperature of these gases after the time t.

E, the mechanical equivalent of heat.

c, specific heat under constant volume of the products of combustion.

ε, the weight of gas formed by the combustion of unit weight of powder.

Let us now consider at any given time t, the change that takes place in the increment of time dt.

During the time dt, the weight y increases by dy, and the quantity of gas for the whole charge increases by ϵdy. The temperature is lowered from T_0 to that of the mixture T, gives off a quantity of heat dq, the expression for which is found by multiplying ϵdy by the specific heat under constant volume c, and by the difference of temperatures $T_0 - T$; hence

$$dq = c\epsilon(T_0 - T)dy. \tag{37}$$

In the same time dt, the quantity ϵy of gas previously formed absorbs the heat dq, produces the external work $d\mathfrak{S}$, and the temperature T varies by dT. Between the quantities dq, $d\mathfrak{S}$, and dT we have the following relation:

$$dq = c\epsilon y dT + \frac{d\mathfrak{S}}{E} \tag{38}$$

[see equation (8), chapter II], recollecting that pdv represents the external work done, that $\frac{R}{c' - c}$ is equal to E, the mechanical equivalent of heat, and that the formula applies to the weight of gas ϵy.

Substituting the value of dq from (37) in (38), we have

$$\frac{d\mathfrak{S}}{E} = c\epsilon(T_0 - T)\,dy - c\epsilon y dT,$$

or

$$d\mathfrak{S} = Ec\,.\,d[\epsilon y\,(T_0 - T)].$$

Integrating from the origin of combustion, we get

$$\mathfrak{S} = Ec\epsilon y\,(T_0 - T). \tag{39}$$

We may now introduce in place of the temperature T of the gases, their volume and pressure on which they directly depend.

By Mariotte's and Gay-Lussac's laws we have

$$pV = R\epsilon y T. \tag{39½}$$

Substituting the value of T from this equation in (39), we have

$$\mathfrak{S} = Ec\epsilon y\left(T_0 - \frac{pV}{R\epsilon y}\right). \tag{40}$$

We now must express V as a function of the motion of the projectile. V being the volume occupied by the gas at the time t, is, in fact, equal to the following:

$$V = \omega\,(z + u),$$

in which

ω = the right section of the bore,

u = travel of the projectile from the origin of motion,

z = *the reduced length of the initial air space*, or is the height of a cylinder whose cross section is ω, the cross section of the bore, and whose volume is equal to the volume of the initial air space V_0.

Putting the value of V in (40), we have

$$\mathfrak{S} = Ec\varepsilon y\left(T_0 - \frac{p\omega(z+u)}{R\varepsilon y}\right). \tag{41}$$

In this equation $p\omega$ is the pressure exerted on the base of the projectile, and if for the present we neglect passive resistances, we have

$$p\omega = m\,\frac{d^2u}{dt^2}, \tag{a}$$

in which m is the mass of the projectile.

Again, if we take no account of the inertia of the charge, and the living forces of the charge, the gun, and the carriage as compared to the much greater living force of the projectile, we have also that the external work $\mathfrak{S}$ is simply equal to $\frac{1}{2}$ the living force of the projectile,

or
$$\mathfrak{S} = \frac{m}{2}\left(\frac{du}{dt}\right)^2. \tag{b}$$

Equation (41) then becomes

$$\frac{m}{2}\left(\frac{du}{dt}\right)^2 = Ec\varepsilon y T_0 - \frac{Ec}{R}\,m\,\frac{d^2u}{dt^2}\,(z+u). \tag{42}$$

This equation may be simplified as follows: We have $E = \dfrac{R}{c'-c}$ and $n = \dfrac{c'}{c}$, and if in equation ($39\frac{1}{2}$) we let $V = 1$, $y = 1$, and $T = T_0$, we have for the corresponding value of p the pressure exerted by the gases of unit weight of powder. When this gas occupies at the temperature of combustion a volume equal to unity, this pressure is equal to that which we have already defined as f *the force of the powder*, or $f = R\varepsilon T_0$, equation (42) then becomes

$$(z+u)\,\frac{d^2u}{dt^2} + \frac{n-1}{2}\left(\frac{du}{dt}\right)^2 = \frac{fy}{m}. \tag{43}$$

In establishing this equation we have neglected:

1. The effect of the cooling of the gas by contact with the walls of the gun;

2. The energies imparted to the charge and the gun and carriage, also the energy of rotation of the projectile.

3. The effect of passive resistances.

The first may be taken into account by a suitable variation of the value of f. The energies imparted to the gun and carriage, and of rotation of the projectile, are too small in comparison with that of the projectile to be taken into account at all. Not so, however, with that imparted to the charge remaining unburned after the time t, and in some way this must be allowed for. We may allow for it by a suitable augmentation of the mass m of the projectile, and the same remark will apply to the passive resistances which have to be overcome. It would amount to the same thing, however, if, instead of changing m to allow for these necessary corrections, we change the quantity f which we call the *force of the powder*.

From this it results that the theoretical value which we have deduced for f is of little use for practical work with the formulas, because we have to take account, in the case of guns, of the various sources of loss of energy.

However, f can only be considered as a coefficient in the formulas which express the motion of projectiles in the bore, and which is determined finally by experiments with guns. What is more important is to establish the *form* of the relations that are obtained with the variable elements of firing, and to determine afterwards the values of such constants as may be necessary from actual firing.

47. Our problem is reduced, then, to finding an integral of equation (43) which will, as well as $\frac{du}{dt}$, the first derivative, vanish for $t=0$.

We have first to find, however, the form of the function y which represents the weight of the charge burned after the time t.

In chapter V the form of function y has already been determined as follows:

$$y=\varpi\psi(t),$$

in which
$$\psi(t)=\frac{at}{\tau}\left(1-\lambda\frac{t}{\tau}+u\frac{t^2}{\tau^2}+\ldots\right),$$

$$\therefore \quad y=\frac{\varpi at}{\tau}\left(1-\lambda\frac{t}{\tau}+u\frac{t^2}{\tau^2}+\ldots\right),$$

and (43) becomes

$$(z+u)\frac{d^2u}{dt^2}+\frac{n-1}{2}\left(\frac{du}{dt}\right)^2=\frac{f\varpi at}{\tau}\left(1-\lambda\frac{t}{\tau}+u\frac{t^2}{\tau^2}+\ldots\right), \quad (44)$$

and in this equation we have supposed that the combustion takes place in free air or under atmospheric pressure, which is evidently not true in the case of the combustion of powder in guns.

48. We must now take account of the variable pressure under which the powder burns. It must be admitted also that the velocity of combustion is proportional to some positive power of the pressure, that is to say, increases with the pressure.

So, let v_0 be the velocity of combustion in free air, or under the normal atmospheric pressure p_0, and v the velocity of combustion under the pressure p; then

$$v = v_0 \left(\frac{p}{p_0}\right)^a.$$

If the pressure remained constant and equal to the atmospheric pressure, the thickness burned in the time t would be $v_0 t$; but the pressure is not constant, and the thickness burned is

$$\int_0^t v\,dt,$$

whence from the value of v preceding we have for the thickness burned at the end of the time t,

$$v_0 \int_0^t \left(\frac{p}{p_0}\right)^a dt.$$

Therefore, in taking into account the variable pressure under which the powder burns we must replace t in equation (44) by the above expression. The value of $y = \varpi\psi(t)$ then becomes, recollecting that $p = \frac{m}{\omega}\left(\frac{d^2u}{dt^2}\right)$,

$$y = \varpi \frac{a}{\tau}\left(\frac{m}{\omega p_0}\right)^a \int_0^t \left(\frac{d^2u}{dt^2}\right)^a dt \left\{ 1 - \frac{\lambda}{\tau}\left(\frac{m}{\omega p_0}\right)^a \int_0^t \left(\frac{d^2u}{dt^2}\right)^a dt + \ldots \right\}.$$

Substituting this value of y in (43), we get the following as a final value of (44):

$$\left.\begin{aligned} (z+u)\frac{d^2u}{dt^2} + \frac{n-1}{2}\left(\frac{du}{dt}\right)^2 = \frac{fa\varpi}{m\tau}\left(\frac{m}{\omega p_0}\right)^a \int_0^t \left(\frac{d^2u}{dt^2}\right)^a dt \\ \times \left\{ \begin{aligned} & 1 - \frac{\lambda}{\tau}\left(\frac{m}{\omega p_0}\right)^a \int_0^t \left(\frac{d^2u}{dt^2}\right)^a dt \\ & + \frac{u}{\tau^2}\left(\frac{m}{\omega p_0}\right)^{2a} \left[\int_0^t \left(\frac{d^2u}{dt^2}\right)^a dt\right]^2 + \ldots \end{aligned} \right. \end{aligned}\right\} \quad (45)$$

The problem is now reduced to one of pure analysis, which consists of finding a function u of t which will satisfy the above equation and

which will cause the equation and its first derivative to vanish for $t = 0$.

We shall see in the next chapter how this may be accomplished.

49. To Find the Value of Z, the Reduced Length of the Air Space.—Since we have supposed all the grains ignited over their whole surfaces simultaneously, the first gas produced expands instantly throughout the interstices of the grains, and before the projectile moves, occupies a volume equal to the initial air space. This volume is as follows:

Let u_0 = the reduced length of the powder chamber, δ the density of the powder, ϖ the weight of the charge, and ω the cross section of the bore; then

$$V_0 = \omega z = \omega u_0 - \frac{\varpi}{\delta},$$

$$z = u_0 - \frac{\varpi}{\omega\delta}.$$

But $\varDelta$, the density of loading, is equal to

$$\frac{\varpi}{\omega u_0} \quad \therefore \quad \frac{\varpi}{\omega\delta} = \frac{\varDelta u_0}{\delta},$$

$$\therefore \quad z = u_0 - \frac{\varDelta}{\delta} u_0 = u_0\left(1 - \frac{\varDelta}{\delta}\right). \qquad (46)$$

This may also be written

$$z = \frac{\varpi}{\omega}\left(\frac{1}{\varDelta} - \frac{1}{\delta}\right). \qquad (47)$$

EXAMPLES.

1. The volume of the chamber of the VI-inch B. L. R. (South Boston) is 920 cubic inches; find the ratio of the volume of the solid powder to that of the chamber when it is loaded with 25 pounds of powder of $\delta = 1.75$.

We find $\varDelta = .752$; whence ratio required $= .43$, about.

2. The largest charge the above gun ever held was 35 pounds of $\delta = 1.75$. Find the ratio of the volume of the solid powder to that of the chamber in this case. We find $\varDelta = 1.053$. *Ans.* .60, about.

3. A powder chamber having a capacity of 1100 cubic inches is loaded with 43 pounds of powder of $\delta = 1.75$. Find the air space.

$\varDelta = 1.082.$

$z\omega = 420.5$ cubic inches.

4. The capacity of the powder chamber of the VI-inch B. L. R., mark III, is 1426 cubic inches; what is the initial air space when the gun is loaded with 54 pounds of powder of density 1.818?

Ans. 618.1 cubic inches.

5. Find the reduced length of the initial air space for the VIII-inch B. L. R., capacity of powder chamber 3824 cubic inches, when loaded with 125 pounds of powder of density 1.867.

Ans. 76.08 inches.

6. A sphere of gunpowder weighing one gram is put in a cubic centimetre and fired. It being assumed that the walls of the envelope are unyielding, find in how long a time it will be completely consumed, if the speed of burning follows Sarrau's law,

$$u = u_0 \left(\frac{p}{p_0}\right)^a. \tag{1}$$

Let ρ be the weight of the powder per unit volume, w the weight burned after any time from the origin, R its primitive, and r any subsequent radius, then

$$w = \frac{4\pi\rho}{3}(R^3 - r^3). \tag{2}$$

Let the initial pressure in the cube be one atmosphere (p_0); then, as the sphere burns, the pressure will depend upon the weight burned, following the law

$$p = p_0 + kw,$$

or

$$p = p_0 + (f - p_0)\,w; \tag{3}$$

since when $w = 1$, $p = f$, the *force* of the powder. Therefore, inserting the value of w in (3), and substituting in (1), we have,

$$\frac{dr}{dt} = -u = -u_0\left[\frac{p_0 + (f-p_0)\frac{4\pi\rho}{3}(R^3 - r^3)}{p_0}\right]^a \tag{4}$$

This gives the solution of the problem. For its numerical solution, taking centimetres, grams, and seconds as units, we have

$$\frac{4\pi\rho}{3}R^3 = 1;$$

whence, if we make $R = \frac{1}{2}$, so that the sphere will just go in, $\rho = 1.9098$. When expressed in atmospheres, we have $p_0 = 1$, $f = 6333$ (Noble and Abel, p. 180). Sarrau takes $a = \frac{1}{2}$; and $u_0 = 1$ cent. is a fair value. Making these substitutions, we have from (4),

$$\frac{dr}{dt} = -79.58\,(1 - 8r^3)^{\frac{1}{2}}.$$

Developing the expression under the radical by the binomial theorem, and retaining 5 terms, we find, calling T the whole time of burning: $T = .0077$ sec.

7. From Example 6, derive the law connecting the time with the radius of a sphere when burned in a closed cylinder in a density of loading of unity.

Retaining three terms only and taking the density as before, we have

$$t = \frac{1}{79.58}\left[R - r + (R^4 - r^4) + \frac{24}{7}(R^7 - r^7)\right].$$

8. In Noble and Abel's record of their experimental work (Table X), it will be found that in a 10-inch rifle loaded with a 300-lb. projectile, the maximum pressure of 18 tons per square inch was reached in .0044 second, and that at that instant the shot had moved 6 inches. Assuming that the pressure increased according to the law $p = kt^n$, where p is the pressure, t the time, and k a constant; find, from the data above, the value of the exponent n.

Taking tons, inches, and seconds as units, and calling T the value of t when the maximum is reached, we have

$$m\frac{d^2u}{dt^2} = 18\pi\,25\left(\frac{t}{T}\right)^n,$$

$$\therefore \quad \frac{d^2u}{dt^2} = \frac{2240 \times 32.2 \times 12 \times 18 \times \pi \times 25}{300}\left(\frac{t}{T}\right)^n.$$

Denote by A all of the numerical factor but T^n; then we find by two integrations, calling U the travel when $t = T$,

$$U = \frac{AT^2}{(n+1)(n+2)},$$

$$\therefore \quad n = 2.16, \text{ about.}$$

CHAPTER VII.

GENERAL DISCUSSION AND PROPERTIES OF THE MOTION OF A PROJECTILE IN THE BORE.

50. We have established in the preceding chapters the general form of the equation which gives the motion of a projectile in the bore. This equation is as follows:

$$
\left.\begin{aligned}
(u+z)\frac{d^2u}{dt^2}+\theta\left(\frac{du}{dt}\right)^2 &= \frac{fa\varpi}{m\tau}\left(\frac{m}{\omega p_0}\right)^a\int_0^t\left(\frac{d^2u}{dt^2}\right)^a dt \\
&\times\left\{\begin{aligned}&1-\frac{\lambda}{\tau}\left(\frac{m}{\omega p_0}\right)^a\int_0^t\left(\frac{d^2u}{dt^2}\right)^a dt\\ &+\frac{\mu}{\tau^2}\left(\frac{m}{\omega p_0}\right)^{2a}\left[\int_0^t\left(\frac{d^2u}{dt^2}\right)^a dt\right]^2\\ &+\ldots\end{aligned}\right.
\end{aligned}\right\} \quad (45)
$$

where

u is the displacement of the projectile,
m its mass;
ϖ the weight of the charge of powder;
ω the right section of the bore;
f the force of the powder;
τ the time of burning of a grain under the normal atmospheric pressure p_0;
$a, \gamma, \mu, \ldots$ numerical coefficients depending upon the form of the grain;
z the reduced length of the initial air space, calculated by (46) or (47);
a the exponent of the power to which we have supposed the velocity of combustion of the powder proportional;
θ the numerical coefficient $=\frac{n-1}{2}$.

The problem of the motion of a projectile in a gun is then reduced to finding a function u of t which will satisfy (45), and vanish, with its first derivative $\frac{du}{dt}$, when $t=0$. The numerical determination of this function may be effected by the ordinary methods of approximation used in the integration of differential equations; it requires that

the numerical coefficients a and θ shall be known. We shall return to this in the succeeding chapter. But, without this determination, we may deduce from the form of this equation certain general properties of the function u which, from a practical view, appear worthy of remark.

51. Equation (45) depends upon the variables enumerated in the preceding number, and changes with the dimensions of the piece used and the manner of loading. We shall first show that we may substitute for it a system of *purely numerical* equations, and common, therefore, to all cases.

Let y equal the ratio of u to the reduced length of the initial air space, so that

$$u = yz, \qquad (a)$$

and put, for shortness,

$$K = \frac{fa\varpi}{mz^2\tau}\left(\frac{mz}{\omega p_0}\right)^a, \qquad (b)$$

$$Y = \int_0^t \left(\frac{d^2y}{dt^2}\right)^a dt; \qquad (c)$$

equation (45) then becomes

$$\left.\begin{aligned}(y+1)\frac{d^2y}{dt^2} + \theta\left(\frac{dy}{dt}\right)^2 = KY\Big[1 - \frac{\lambda}{\tau}\left(\frac{mz}{\omega p_0}\right)^a Y \\ + \frac{\mu}{\tau^2}\left(\frac{mz}{\omega p_0}\right)^{2a} Y^2 + \ldots\Big].\end{aligned}\right\} \qquad (d)$$

Change the independent variable t to x, defined by the relation

$$x = K^\beta t, \qquad (e)$$

K being the quantity defined by (b), and β an undetermined exponent. We have

$$\frac{dy}{dt} = K^\beta \frac{dy}{dx}, \quad \frac{d^2y}{dt^2} = K^{2\beta}\frac{d^2y}{dx^2}. \qquad (f)$$

We have also $dt = K^{-\beta}.\,dx$. From this, and the value of $\frac{d^2y}{dx^2}$ in (f), (c) becomes

$$Y = K^{(2a-1)\beta}\int_0^x \left(\frac{d^2y}{dx^2}\right)^a . \, dx, \qquad (g)$$

or

$$Y = K^{(2a-1)\beta} X, \qquad (h)$$

if we write

$$X = \int_0^x \left(\frac{d^2y}{dx^2}\right)^a . \, dx. \qquad (i)$$

Substituting the values from (f) and (h) in (d), and putting

$$\gamma = 1 + (2a - 3)\beta, \; \varepsilon = \left(\frac{mz}{\omega p_0}\right)^a K^{(2a-1)\beta}, \qquad (j)$$

we find

$$(y+1)\frac{d^2y}{dx^2}+\theta\left(\frac{dy}{dx}\right)^2=K^\gamma X(1-\lambda\varepsilon X+\mu\varepsilon^2X^2+\ldots) \qquad (k)$$

Determining β so as to make γ vanish, which, by (j), gives

$$\beta=\frac{1}{3-2a}, \qquad (l)$$

$$\varepsilon=\frac{1}{\tau}\left(\frac{mz}{\omega p_0}\right)^a K^{\frac{2a-1}{3-2a}}, \qquad (m)$$

equation (k) is finally reduced to the following,

$$(y+1)\frac{d^2y}{dx^2}+\theta\left(\frac{dy}{dx}\right)^2=X-\lambda\varepsilon X^2+\mu\varepsilon^2X^3+\ldots \qquad (n)$$

52. To satisfy this equation, suppose the unknown function y developed in a series of ascending powers of ε, in the form

$$y=y_0+\varepsilon y_1+\varepsilon^2y_2+\ldots, \qquad (o)$$

$y_0, y_1, y_2, \ldots$ being unknown functions of x. We have then

$$\frac{d^2y}{dx^2}=\frac{d^2y_0}{dx^2}\left[1+\varepsilon\frac{d^2y_1}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)^{-1}+\varepsilon^2\frac{d^2y_2}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)^{-1}+\ldots\right].$$

Developing $\left(\frac{d^2y}{dx^2}\right)^a$ by the binomial theorem, and putting

$$\left.\begin{aligned}
X_0&=\int_0^x\left(\frac{d^2y_0}{dx^2}\right)^a.\,dx,\\
X_1&=a\int_0^x\frac{d^2y_1}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)^{a-1}.\,dx,\\
X_2&=a\int_0^x\frac{d^2y_2}{dx^2}\left(\frac{d^2y_0}{dx^2}\right)^{a-1}.\,dx\\
&\quad+\frac{a(a-1)}{1.2}\int_0^x\left(\frac{d^2y_1}{dx^2}\right)^2\left(\frac{d^2y_0}{dx^2}\right)^{a-2}.\,dx\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \qquad (p)$$

we easily find for (i),

$$X=X_0+\varepsilon X_1+\varepsilon^2X_2+\ldots \qquad (q)$$

Substituting the values of y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, X, in (n), and equating the coefficients of like powers of ε, we have the following system of equations,

$$\left.\begin{array}{l}
(y_0+1)\dfrac{d^2y_0}{dx^2}+\theta\left(\dfrac{dy_0}{dx}\right)^2=X_0,\\[2ex]
(y_0+1)\dfrac{d^2y_1}{dx^2}+2\theta\,\dfrac{dy_0}{dx}\cdot\dfrac{dy_1}{dx}+\dfrac{d^2y_0}{dx^2}y_1=X_1-\lambda X_0^2,\\[2ex]
\left.\begin{array}{l}(y_0+1)\dfrac{d^2y_2}{dx^2}+2\theta\dfrac{dy_0}{dx}\cdot\dfrac{dy_2}{dx}\\ \quad+\dfrac{d^2y_0}{dx^2}y_2+y_1\dfrac{d^2y_1}{dx^2}+\theta\left(\dfrac{dy_1}{dx}\right)^2\end{array}\right\}=X_2-2\lambda X_0X_1+\mu X_0^3
\end{array}\right\}\quad(r)$$

and it will fulfill all the conditions of the problem if we find a system of functions $y_0, y_1, y_2, \ldots$, which will satisfy these equations, and vanish with their first derivatives when $x=0$.

These equations are numerical, and remain the same whatever may be the conditions of fire, for the same system of values for λ, μ, . . . , that is to say, for the same form of grain.

The functions which are determined by these equations may be tabulated, as in the case of logarithms and the circular functions; and this labor performed, we can see that (o) furnishes the complete solution of the problem.

53. General Formulæ of Initial Velocities and Pressures. —The displacement of the projectile u being equal to zy, the velocity v and acceleration w have the values, from (f),

$$v=zK^{\beta}\frac{dy}{dx},\quad w=zK^{2\beta}\frac{d^2y}{dx^2},\qquad (s)$$

because

$$v=\frac{du}{dt}=\frac{du}{dx}\,\frac{dx}{dt}=z\,\frac{dy}{dx}\,\frac{dx}{dt}=zK^{\beta}\,\frac{dy}{dx},$$

$$w=\frac{d^2u}{dt^2}=zK^{\beta}\,\frac{d^2y}{dx^2}\,\frac{dx}{dt}=zK^{2\beta}\,\frac{d^2y}{dx^2}.$$

To deduce the pressure p on unit surface, we have only to multiply w by m and divide by ω. If we replace y by its value (o), we have the two formulæ

$$v=zK^{\beta}\left(\frac{dy_0}{dx}+\varepsilon\,\frac{dy_1}{dx}+\varepsilon^2\,\frac{dy_2}{dx}+\ldots\right),\qquad (t)$$

$$p=\frac{mz}{\omega}\,K^{2\beta}\left(\frac{d^2y_0}{dx^2}+\varepsilon\,\frac{d^2y_1}{dx^2}+\varepsilon^2\,\frac{d^2y_2}{dx^2}+\ldots\right).\qquad (u)$$

The coefficients of the second member are numerical coefficients of $x=K^{\beta}t$. These formulæ express, then, the velocity and pressure as a function of the time. To express them as a function of the distance passed over, we have only to suppose that we have taken

from (o) the value of x as a function of y in the form

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots \qquad (v)$$

and substituted this value in (t) and (u), whose coefficients then become functions of $y = \frac{u}{z}$. We have thus

$$v = zK^\beta \left[F_0\left(\frac{u}{z}\right) + \varepsilon F_1\left(\frac{u}{z}\right) + \varepsilon^2 F_2\left(\frac{u}{z}\right) + \dots \right] \qquad (48)$$

$$p = \frac{mz}{\omega} K^{2\beta} \left[f_0\left(\frac{u}{z}\right) + \varepsilon f_1\left(\frac{u}{z}\right) + \varepsilon^2 f_2\left(\frac{u}{z}\right) + \dots \right], \qquad (49)$$

which show the forms of the relations which must connect v and p with the various elements of fire. The functions F and f are numerical; the quantities K, ε, β have the values given in (b), (m), (l).

By giving to u a value equal to the initial distance of the base of the projectile from the muzzle, (48) evidently becomes a formula for initial velocity.

54. Similar Guns.—Two guns are *similar* when all their homologous linear dimensions are proportional to their calibres. The reduced length of the chamber being then proportional to the calibre, the quantity z, which is itself proportional to the former, varies in the same ratio; the cross section of the bore ω is proportional to the square of the calibre.

The similitude is extended to the loading when the weights of the powder and the projectile are proportional to the cube of the calibre, and when the grains of powder have the same form and dimensions proportional to the calibre. Consequently, the coefficients a, λ, μ, . . . must have the same value, and the time of combustion τ must vary, between the two guns, proportionally to the calibre.

When these conditions are fulfilled the relations (b) and (m) show that K^β varies in inverse ratio of the calibre, and that ε is constant. It results, therefore, from (48) and (49) that the values of v and p are equal for equal values of $\frac{u}{z}$, that is, for values of u proportional to the calibre.

This principle follows:

In similar guns, similarly loaded, the velocities and pressures corresponding to distances passed over proportional to the calibres, are equal.

This principle requires that the duration of combustion shall be proportional to the calibre; it ceases to be applicable if, all other

circumstances of similitude being observed, the powder is the *same* in the two guns. The equality of the velocities and pressures, then, no longer exists, contrary to the opinion which is very generally held. Consequently, the empirical formulæ, established upon the hypothesis of this equality, are not exact; we shall see further on that the error committed by their use, even in guns which are nearly alike, may be quite large.

55. The principle of the similitude of guns being of great importance in practical applications, it has been thought better to deduce them from the fundamental formulæ in their most general form. In the researches which follow, we shall simplify the formulæ by limiting the number of terms considered. The quantity ϵ being generally quite small, we may admit that the series (48) and (49) are very convergent, and that the first two terms give a sufficient approximation, similarly to what we have already stated in our first researches, in deriving the expression which gives the initial velocity when the effect of the variation of pressure upon the combustion of the powder is neglected.

The formulæ are thus greatly simplified, and we may, moreover, introduce, in explicit form, the coefficient λ, which, as well as a, depends upon the form of the grain.

56. Consider the development (o) which expresses the displacement of the projectile as a function of the time. The first term, y_0, is defined by the first of (r), which, as has already been remarked, is purely numerical, and does not depend on either λ or μ . . . The coefficient y_1 of the second term satisfies the second of (r). Replacing X_1 in this equation by its value in (p), we easily see that all the terms containing the unknown quantity are *linear*. Moreover, the term which is independent of the unknown quantity, being proportional to λ, it is clear that y_1 is the product of λ by a numerical function of x.

The second term of (o), and consequently the second term of (v), are then proportional to λ. We have the same results also evidently for the developments (48) and (49), whose second terms are equal to the product of λ by functions of $\frac{u}{z}$.

These functions are, moreover, positive. In fact, if the combustion of a grain under constant pressure were uniform, the coefficients $\lambda, \mu, \ldots$, would vanish, and the series (48) and (49) would be reduced rigidly to their first terms. But the combustion is not uniform in

possible shapes of grain; on account of the progressive diminution of the surfaces of emission, the gas is formed with a decreasing velocity; this causes a diminution of the initial velocity and of the rate of increase of pressure in the space passed over. We may then put

$$F_1\left(\frac{u}{z}\right)=-\lambda\varphi_1\left(\frac{u}{z}\right), f_1\left(\frac{u}{z}\right)=-\lambda\psi_1\left(\frac{u}{z}\right),$$

φ_1, ψ_1 being increasing functions of $\frac{u}{z}$. If, to make the notation symmetrical, we write φ_0 and ψ_0 in place of F_0 and f_0, formulæ (48) and (49), reduced to their first terms, become

$$v=zK^\beta\left[\varphi_0\left(\frac{u}{z}\right)-\varepsilon\lambda\varphi_1\left(\frac{u}{z}\right)\right], \tag{50}$$

$$p=\frac{mz}{\omega}K^{2\beta}\left[\psi_0\left(\frac{u}{z}\right)-\varepsilon\lambda\psi_1\left(\frac{u}{z}\right)\right], \tag{51}$$

whence we derive several important consequences.

57. Formula for Pressures.—Consider first (51), which gives the pressure.

The maximum pressure, that which endangers the gun, occurs generally when the shot has moved a small distance only; the value of p may at that point be reduced to that of its first term. We have then

$$p=\frac{mz}{\omega}K^{2\beta}\psi\left(\frac{u}{z}\right). \tag{52}$$

The function ψ is purely numerical, and entirely independent of the elements of fire. Let b be the value of the variable which corresponds with the maximum, and let A be the value of the maximum. The maximum pressure in the bore is then

$$P=A\frac{mz}{\omega}K^{2\beta}, \tag{53}$$

or, recollecting the values of K and β in (b) and (l),

$$P=A\frac{mz}{\omega}\left(\frac{fa\varpi}{\tau m z^2}\right)^{\frac{2}{3-2a}}\left(\frac{mz}{\omega p_0}\right)^{\frac{2a}{3-2a}}. \tag{54}$$

Expressed in atmospheres, the pressure is

$$\frac{P}{p_0}=A\left(\frac{mz}{\omega p_0}\right)^{\frac{2a}{3-2a}}\left(\frac{fa\varpi}{\tau m z^2}\right)^{\frac{2}{3-2a}}. \tag{55}$$

58. We have called b the value of the variable for which $\psi\left(\frac{u}{z}\right)$ is a maximum. The corresponding value of the displacement is

$$u=bz.$$

We therefore see that, in the degree of approximation adopted, the displacement of the projectile corresponding to the maximum pressure is proportional to the *reduced length of the initial air space.* The only elements of fire upon which its value depends are then, by (46), the reduced length of the chamber, the density of loading, and the density of the powder.

This depends upon the supposition that the expression for the pressure is reduced to its first term. It ceases to be the case if the second term may not be neglected. This may become the case from the use of rapidly burning powders; the value of ϵ, (m), increases as τ decreases.

The value of $y = \frac{u}{z}$, corresponding to the maximum of p, may be obtained by putting the first derivative of $\psi_0(y) - \epsilon\lambda\psi_1(y)$ equal to zero. We thus have

$$\psi'_0(y) - \epsilon\psi'_1(y) = 0.$$

Let $b + \delta b$ be the root we wish to find; we have then

$$\psi'_0(b + \delta b) - \epsilon\lambda\psi'_1(b + \delta b) = 0.$$

Considering δb a quantity of the first order, noting that $\psi'(b) = 0$, and neglecting the quantities of the second order, we have

$$\delta b = \epsilon\lambda \frac{\psi'_1(b)}{\psi'_0(b)};$$

$\psi''_0(b)$ is negative because $\psi_0(b)$ is a maximum, $\psi'_1(b)$ is positive because ψ_1 is an increasing function; therefore δb is negative. Consequently the displacement corresponding to the maximum pressure is less than bz, which is therefore a superior limit.

The maximum pressure is a little less than that given by (53). In fact, for the value $b + \delta b$ of the variable, the function $\psi_0(y) - \epsilon\lambda\psi_1(y)$ is reduced, by neglecting the small quantities of the second order, to

$$\psi_0(b) - \epsilon\lambda\psi_1(b).$$

This value is less than $\psi_0(b) = A$. In fact, the influence of the second term seems generally small, and (53) and (55) give the pressure with a sufficient approximation for practical purposes, as we shall see later.

59. Formula for Velocity.—On the contrary, the two first terms must be retained for the initial velocity, which corresponds nearly always to a value of $\frac{u}{z}$ decidedly larger than the one corresponding to the maximum pressure. Upon the ratio between these two terms

depends the influence of the form of grain upon the relative values of the velocities and pressures. We shall treat this matter more at length. The formula for velocity, reduced to two terms, when K and ε are replaced by their values, becomes

$$\left.\begin{aligned} v = z\left(\frac{fa\varpi}{\tau m z^2}\right)^{\frac{1}{3-2a}}\left(\frac{mz}{\omega p_0}\right)^{\frac{a}{3-2a}}\Bigg[\varphi_0\left(\frac{u}{z}\right) \\ -\frac{\lambda}{\tau}\left(\frac{fa\varpi}{\tau m z^2}\right)^{\frac{2a-1}{3-2a}}\left(\frac{mz}{\omega p_0}\right)^{\frac{2a}{3-2a}}\varphi_1\left(\frac{u}{z}\right)\Bigg]. \end{aligned}\right\} \quad (56)$$

60. This formula is somewhat complex, but may be much simplified by putting $a = \frac{1}{2}$, and the following considerations lead to the adoption of this value, although we do not state it as exact.

The movement of the inflaming gases penetrating into the interior of the substance of the powder can be considered as placed under the action of two opposing forces, which are the exterior pressure p, and the resistance of the material, which, arising from the movement of the gases, is a function $f(v)$ of their velocity.

These two forces are in equilibrium, since the movement of the gases is uniform under constant pressure, and we have the relation

$$p = f(v).$$

This being granted, the resistance $f(v)$, reduced to the unit of mass of the gases, is probably proportional:

1st. To the velocity v, that is to say, to the number of particles of the resisting medium which are opposed in the unit of time to the movement of the gases.

2d. To a certain function of the velocity vanishing for $v = 0$, and proportionally representing the resistance due to each particle of the medium.

In admitting that this last function should be proportional to the velocity, the function $f(v)$ would be proportional to the square of the velocity, and it follows from the relation $p = f(v)$, the velocity would be *proportional to the square root of the pressure.*

61. Formula for Velocities in the Hypothesis, $a = \frac{1}{2}$.—Putting $a = \frac{1}{2}$ in (56), we find

$$v = \left(\frac{fa\varpi}{\tau m}\right)^{\frac{1}{2}}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{4}}\left[\varphi_0\left(\frac{u}{z}\right) - \frac{\lambda}{\tau}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{2}}\varphi_1\left(\frac{u}{z}\right)\right]. \quad (57)$$

To apply this formula, the functions φ_0 and φ_1 must be known, or (r) must be integrated. In this formula we designate by

ϖ the weight of the charge of powder,
m the mass of the projectile,
u the length of the path of the projectile (this length is the distance which separates, at the position of loading, the base of the projectile from the face of the muzzle. This is the length, and not the total length of the bore, which directly influences the velocity),
ω the right section of the bore,
z the reduced length of the initial air space. (*Initial air space* is the difference between the volume of the powder chamber and the volume of the powder. The *reduced length* corresponding is measured by the height of a cylinder having for a base the right section of the bore and for a volume that of the *initial air space*),
f the force of the powder. (The force of a powder is the pressure of the gases of 1 kilogram of that powder occupying, at the temperature of combustion, the unit of volume. See Art. 61, Part II),
p_0 the normal atmospheric pressure,
τ the duration of the combustion of a grain of powder under the pressure p_0,
α, λ numerical coefficients depending upon the form of the grain (these values for various forms of grains will be found, Articles 39, 40, 41, Part III);
φ_0, φ_1 are purely numerical functions.

The reduced length of the *initial air space* is calculated by formula

$$z = \frac{\varpi}{\omega}\left(\frac{1}{\Delta} - \frac{1}{\delta}\right),$$

in which we call
Δ the density of loading,
δ the real density of the powder.

62. The theoretical determination of the functions φ_0 and φ_1 offers no special difficulties. But this determination is not indispensable for the applications which we have in view, and we can, as will be seen, deduce from the relation (57) a formula which suffices for the wants of practice.

63. Whatever may be the analytical expressions for φ_0 and φ_1, these functions are such that they become respectively proportional to the $\frac{1}{2}$ and $\frac{3}{4}$ powers of $\frac{u}{z}$ for infinitely great values of that variable.

This condition results from the analytical discussion of the equations which define these auxiliary functions.

Also we observe that the two terms of expression (57), respectively proportional to $z^{\frac{1}{2}}$ and $z^{\frac{3}{4}}$, remain finite for infinitely small values of z.

So that we have, when $\frac{u}{z}$ is very great,

$$\varphi_0\left(\frac{u}{z}\right)=A_0\left(\frac{u}{z}\right)^{\frac{1}{4}},\ \varphi_1\left(\frac{u}{z}\right)=A_1\left(\frac{u}{z}\right)^{\frac{3}{4}}, \qquad (58)$$

A_0 and A_1 representing numerical coefficients.

This being granted, we write formula (57) as follows :

$$v=\left(\frac{fa\varpi}{\tau m}\right)^{\frac{1}{2}}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{4}}\varphi_0\left(\frac{u}{z}\right)\left[1-\frac{\lambda}{\tau}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{2}}\frac{\varphi_1\left(\frac{u}{z}\right)}{\varphi_0\left(\frac{u}{z}\right)}\right]. \qquad (59)$$

In the ordinary conditions of practice, the ratio $\frac{u}{z}$ has a large value; on the other hand, the second term of the quantity between brackets is generally a fraction much smaller than unity. We can, therefore, substitute approximately for the ratio $\frac{\varphi_1}{\varphi_0}$ the value which it approaches for increasing values of $\frac{u}{z}$. From equation (58) we have then very nearly

$$\frac{\varphi_1\left(\frac{u}{z}\right)}{\varphi_0\left(\frac{u}{z}\right)}=Q\left(\frac{u}{z}\right)^{\frac{1}{2}}.$$

Admitting this hypothesis, and supposing, further, that the function φ_0 is sensibly proportional within certain limits to a power γ of the variable, that is to say, putting

$$\varphi_0\left(\frac{u}{z}\right)=P\left(\frac{u}{z}\right)^{\gamma},$$

it becomes

$$v=P\left(\frac{fa\varpi}{\tau m}\right)^{\frac{1}{2}}\left(\frac{mz}{\omega p_0}\right)^{\frac{1}{4}}\left(\frac{u}{z}\right)^{\gamma}\left[1-Q\frac{\lambda}{\tau}\left(\frac{mu}{\omega p_0}\right)^{\frac{1}{2}}\right],$$

or better still, replacing z by its value,

$$\left.\begin{array}{l} v=Pp_0^{-\frac{1}{4}}\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\varpi}{m}\right)^{\frac{1}{4}}\left(\frac{\varpi}{\omega}\right)^{\frac{1}{4}-\gamma} \\ \left(\frac{1}{\Delta}-\frac{1}{\delta}\right)^{\frac{1}{4}-\gamma}u^{\gamma}\left[1-Q\frac{\tau}{\lambda}\left(\frac{mu}{\omega p_0}\right)^{\frac{1}{2}}\right]. \end{array}\right\} \qquad (60)$$

64. Designating by p the weight of the projectile, c the calibre or

diameter of the bore, g the acceleration of the force of gravity, we have

$$m = \frac{p}{g}, \quad \omega = \frac{\pi c^2}{4}.$$

Substituting these values in expression (60), and putting for brevity

$$A = P\left(\frac{g}{p_0}\right)^{\frac{1}{4}}\left(\frac{4}{\pi}\right)^{\frac{1}{2}-\gamma}, \quad B = Q\left(\frac{4}{\pi g p_0}\right)^{\frac{1}{2}},$$

it becomes

$$v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\varpi}{p}\right)^{\frac{1}{4}}\left(\frac{\varpi}{c^2}\right)^{\frac{1}{2}-\gamma}\left(\frac{1}{\Delta}-\frac{1}{\delta}\right)^{\frac{1}{4}-\gamma} u^{\gamma}\left(1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right). \quad (61)$$

65. In order to obtain a formula which shall be numerically applicable, it remains to fix the value of γ. It can be obtained from the results of experiments. In fact, the empirical formulæ of the commission of Gavre established the fact that, all other elements being constant, the initial velocity is:

1st. Proportional to the $\frac{5}{8}$ power of the charge.

2d. Inversely proportional to the $\frac{1}{4}$ power of the capacity S in which the powder charge is placed.

Now, if we call S the capacity of the powder chamber, we have $\Delta = \frac{\varpi}{S}$; consequently (61) becomes

$$v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\frac{\varpi^{\frac{1}{4}+\gamma}S^{\frac{1}{2}-2\gamma}u^{\gamma}}{p^{\frac{1}{4}}c^{1-2\gamma}\delta^{\gamma-\frac{1}{4}}}\left(1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right).$$

If, now, we give the exponent of ϖ the value $\frac{5}{8}$, the value of γ is found to be $\frac{3}{8}$, and the exponent of S becomes $-\frac{1}{4}$, which accords with practice. Substituting the value $\gamma = \frac{3}{8}$ in (61), and allowing the factor $\frac{1}{\delta^{\frac{1}{8}}}$, the variations of which are small, to be included in the factor A, we have

$$v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}(\varpi u)^{\frac{3}{8}}\left(\frac{\Delta}{pc}\right)^{\frac{1}{4}}\left(1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right), \quad (62)$$

in which $u =$ the travel of the projectile according to the notation adopted by Sarrau.

66. The Formula for Maximum Pressure.—The maximum pressure is generally produced when the projectile has travelled a comparatively short distance along the bore. The value of P, as we have seen, equations (54) and (55), is

$$P = A\frac{mz}{\omega}\left(\frac{fa\varpi}{\tau m z^2}\right)^{\frac{2}{3-2a}}\left(\frac{mz}{\omega p_0}\right)^{\frac{2a}{3-2a}}, \quad (54)$$

or when expressed in atmospheres,

$$P = A\left(\frac{mz}{\omega p_0}\right)^{\frac{2a}{3-2a}}\left(\frac{fa\varpi}{\tau mz^2}\right)^{\frac{2}{3-2a}}. \tag{55}$$

Putting $a = \frac{1}{2}$ in (54), we have

$$P = A\,\frac{fa\varpi}{\omega\tau}\left(\frac{m}{z\omega p_0}\right)^{\frac{1}{2}}.$$

Now replacing z by its value, $z = \frac{\varpi}{\omega}\left(\frac{1}{\Delta} - \frac{1}{\delta}\right)$, we have

$$P = A\,\frac{fa}{\tau\omega}\left(\frac{m\varpi\delta\Delta}{p_0(\delta - \Delta)}\right)^{\frac{1}{2}}, \tag{63}$$

in which A denotes a numerical factor which is independent of the choice of units. Now put $m = \frac{p}{g}$, in which $p =$ the weight of the projectile, $\omega = \frac{\pi}{4}c^2$, and calling K_1 the product of all the constant factors which enter into the equation, and we have, since

$$K_1 = \frac{A\pi}{g4p_0^{\frac{1}{2}}},$$

$$P = K_1\,\frac{fa}{\tau}\,\frac{(p\varpi)^{\frac{1}{2}}}{c^2}\left(\frac{1}{\Delta} - \frac{1}{\delta}\right)^{-\frac{1}{2}}.$$

67. It is known that a function is sensibly constant near a maximum. Now, in the ordinary conditions of practice, the ratio $\frac{\Delta}{\delta}$ differs but little from $\frac{1}{2}$, which gives to the function $\frac{\Delta}{\delta}\left(1 - \frac{\Delta}{\delta}\right)$ its greatest value, equal to $\frac{1}{4}$. We have, then, nearly,

$$\frac{\Delta}{\delta}\left(1 - \frac{\Delta}{\delta}\right) = \frac{1}{4},$$

and consequently

$$\frac{1}{\Delta} - \frac{1}{\delta} = \frac{1}{4}\,\frac{\delta}{\Delta^2}.$$

Now if we neglect, as we may without error of value, the variations of the square root of δ, the formula for maximum pressure becomes

$$P = K\frac{fa}{\tau}\cdot\frac{\Delta\,(p\varpi)^{\frac{1}{2}}}{c^2}. \tag{64}$$

P is the pressure on the base of the projectile.

It will be noticed that the formula for velocity contains two constants, A and B, and the formula for pressure one constant, K. These constants are for the *same powder*, independent of the elements

of firing. We shall see later how they may be determined by experiment.

68. The Displacement of the Projectile corresponding to the Maximum Pressure.—We have seen that the theoretical study of the motion of a projectile in the interior of a gun should be followed by the determination of certain numerical functions, which may be termed *auxiliaries*, and are designated by the notation $y_0, y_1, y_2 \ldots$, etc., defined by equations (r).

The calculation of these functions is long and laborious and presents no theoretical difficulty. It will not be attempted in this work; for a full discussion and method employed see Proceedings U. S. Naval Institute, No. 28, pages 126 et seq.

69. The function y_0, however, possesses unusual interest, as by its aid we are able to determine the travel of the projectile when the pressure is a maximum.

From the first of equations (r) we have, putting for X_0 its value from the first of equations (p),

$$(y_0 + 1)\frac{d^2y_0}{dx^2} + \theta\left(\frac{dy_0}{dx}\right)^2 = \int_0^x \left(\frac{d^2y_0}{dx^2}\right)^a dx, \qquad (65)$$

in which $\theta = \frac{1}{5}$, about. It should vanish also, as well as its first derivative for the value $x = 0$. For small values of x we satisfy the equation (65) and also the initial conditions by putting

$$\left.\begin{aligned}
&\frac{d^2y_0}{dx^2} = ax^m + bx^n + cx^p + \ldots \\
&\frac{dy_0}{dx} = \frac{a}{m+1}\cdot x^{m+1} + \frac{b}{n+1}\cdot x^{n+1} + \frac{c}{p+1}\cdot x^{p+1} + \ldots \\
&y_0 = \frac{a}{(m+1)(m+2)}\cdot x^{m+2} + \frac{b}{(n+1)(n+2)}\cdot x^{n+2} \\
&\quad + \frac{c}{(p+1)(p+2)}\cdot x^{p+2} + \ldots
\end{aligned}\right\} \qquad (66)$$

$a, b, c, \ldots m, n, p, \ldots$ to be determined. Substituting the values (66) in equation (65), and identifying, we find for the exponents

$$m = \frac{1}{1-a},\ n = \frac{4-2a}{1-a},\ p = \frac{7-4a}{1-a}\ldots;\quad a = \frac{1}{2}; \qquad (67)$$

and the coefficients $a, b, c, \ldots$ are given by the equations

$$\left.\begin{aligned}
&a = \frac{1}{m^m},\\
&\frac{a^2}{(m+1)(m+2)} + b + \theta\frac{a^2}{(m+1)^2} = \frac{am}{n}b,\\
&\frac{ab}{(m+1)(n+2)} + \frac{ab}{(n+1)(n+2)} + c + 2\theta\frac{ab}{(m+1)(n+1)}\\
&\qquad = \frac{am}{p}c + \frac{a(a-1)m}{2p}\cdot\frac{b^2}{a}.
\end{aligned}\right\} \qquad (68)$$

The series (66) are very convergent for values of x less than unity. We can also deduce from them the values of y_0 and of its derivatives for $x = 1$, but beyond $x = 1$ they become rapidly divergent, and it becomes necessary to have recourse to Taylor's Theorem.

Let x_1, an initial value of x, be a value much less than unity.

Formulas (66) give the corresponding values of $\frac{d^2y_0}{dx^2}$, $\frac{dy_0}{dx}$, y_0, and by differentiating several times equations (66) we obtain formulas which allow us to calculate the successive derivatives $\frac{d^3y_0}{dx^3}$, $\frac{d^4y_0}{dx^4}\ldots$, etc., and consequently we will have all the coefficients of the developments

$$\frac{d^2y_0}{dx^2} = \left(\frac{d^2y_0}{dx^2}\right)_{x_1} + \left(\frac{d^3y_0}{dx^3}\right)_{x_1}(x-x_1) + \left(\frac{d^4y_0}{dx^4}\right)_{x_1}\frac{(x-x_1)^2}{2} + \ldots$$

whence, by integrating twice between the limits x_1 and x, we get

$$\frac{dy_0}{dx} = \left(\frac{dy_0}{dx}\right)_{x_1} + \left(\frac{d^2y_0}{dx^2}\right)_{x_1}(x-x_1) + \left(\frac{d^3y_0}{dx^3}\right)_{x_1}\frac{(x-x_1)^2}{2} + \ldots$$

$$y_0 = y_{0_{x_1}} + \left(\frac{dy}{dx}\right)_{x_1}(x-x_1) + \left(\frac{d^2y_0}{d^2x}\right)_{x_1}\frac{(x-x_1)^2}{2} + \ldots$$

And by means of these formulas, suppose the functions $\frac{d^2y_0}{dx^2}$, $\frac{dy_0}{dx}$, y_0 calculated for a given value of x, the variable, as follows:

y_0	$\frac{d^2y_0}{dx^2}$	y_0	$\frac{d^2v_0}{dx^2}$	y_0	$\frac{d^2y_0}{dx^2}$
0.1	0.480	0.6	0.710	1.25	0.651
0.2	0.605	0.7	0.705	1.50	0.621
0.3	0.665	0.8	0.700	1.75	0.590
0.4	0.693	0.9	0.692	2.00	0.563
0.5	0.700	1.0	0.680	2.50	0.513

It is superfluous to prolong this table for greater values of y_0; the influence of the second term which depends on y_1 becomes more and

more sensible, and the value of $\frac{d^2y_0}{dx^2}$ ceases to represent the interior pressure with a sufficient approximation.

We recollect that the variable y_0 represents in the order of approximation adopted, the ratio $\frac{u}{z}$ of the displacement of the projectile to the *reduced length of the initial air space*, which is calculated by the formula

$$z = u_0\left(1 - \frac{\Delta}{\delta}\right),$$

in which we designate by

u_0 the reduced length of the powder chamber;

Δ the density of loading;

δ the density of the powder.

We see by an inspection of the table:

1st. That the maximum of $\frac{d^2y_0}{dx^2}$ is equal to 0.710;

2d. That the corresponding value of y_0 is very nearly 0.6.

And we deduce finally from what precedes a very important result. The maximum pressure is produced when the ratio of the displacement of the projectile to the reduced length of the initial air space is equal to 0.6. Consequently, denoting by U the displacement corresponding to the maximum, we have very nearly,

$$U = 0.6u_0\left(1 - \frac{\Delta}{\delta}\right). \tag{68}$$

On the other hand, this result should be considered as approximate only. But the approximation is sufficient, we believe, in the greater number of cases.

Example. In a 24 centimetre gun, fired under the ordinary conditions of powder proof, we have (the decimetre being taken for the unit): $u_0 = 7.63$, $\Delta = 0.800$, $\delta = 1.800$,

and consequently, $U = 2.54$.

70. This table has another important application, as it enables us to determine to a fair degree of approximation the pressure near the maximum, when we know the latter. We have seen from the table that when the ratio of the travel of the projectile to the reduced length of initial air space is .6, that the pressure is a maximum, evidently within limits; when the travel of the projectile is, for example, $\frac{10}{6}$ of that when the pressure is a maximum, the pressure

is $\frac{68}{71}$ of the maximum pressure. Thus the approximate form of the pressure curve near the maximum may be determined, and is of value in laying down pressure curves in designing guns, or in determining the pressure curve for a given powder in a given gun.

71. The maximum pressure we have spoken of and represented by formula (64) is evidently the maximum pressure on *the base of the projectile* if the constant K is determined by computation.

EXAMPLES.

1. Equation 63 may be written for use with the same powder,

$$P = A_1 \left(\frac{p \varpi \delta \Delta}{\delta - \Delta} \right)^{\frac{1}{2}} \frac{1}{c^2}.$$

Having given the following data of the firing of the 8-inch M. L. R. with sphero-hexagonal powder of 123 granules per pound, all quantities being stated in inches and pounds,

$$\varpi = 35, \qquad \delta = 1.775,$$
$$P = 26600, \qquad \Delta = .923,$$
$$c = 8, \qquad p = 180;$$

it is required to find A_1. $A_1 = 15467$.

2. It is required to apply the result in Example 2 to the 6-inch B. L. R. when loaded with spheres of the same weight, as follows:

$$p = 75.4, \qquad \Delta = .962,$$
$$\varpi = 32, \qquad c = 6.$$
$$\delta = 1.79,$$

$P = 30435$.

The measured pressure was 30975.

3. It is required, from the value of A_1 determined in Example 2, to find the pressure in the 60-pound B. L. R. when loaded with 10 pounds of the powder there described, and a projectile weighing 46.5 pounds. The capacity of the chamber is 308 cubic inches.

$P = 16023$.

The measured pressure was 18300.

4. Prove that if δ increases, and all other elements of loading remain constant, P decreases.

This may be shown from (63).

5. Show that if the form of the grain, weight of projectile, and size of chamber be fixed, (63) may be written in the form

$$P = a \cdot \left(\frac{\delta \Delta^2}{\delta - \Delta} \right)^{\frac{1}{2}};$$

where a is a constant. Show from this, that for small changes, $dP = 0$, when

$$d\Delta = \frac{\Delta^2}{2\delta^2 - \Delta\delta} \cdot d\delta.$$

6. Plot a pressure curve in the neighborhood of its maximum by the table on p. 75.

7. Find the travel of the shot at the instant of maximum pressure in the 8-inch B. L. R. when loaded with 125 pounds of powder of $\delta = 1.867$. The volume of the chamber is 3824 cubic inches.

$U = 23.73$ inches.

8. Find the travel of the shot at the instant of maximum pressure in the 60-pdr. B. L. R. when loaded with 6 pounds of powder of $\delta = 1.79$. The volume of the chamber is 308 cubic inches.

$U = 9.8$ inches.

CHAPTER VIII.

PRACTICAL FORMULAS FOR INITIAL VELOCITY AND MAXIMUM PRESSURE.

72. In the preceding chapter the form of the equations for initial velocity and maximum pressure on the base of the projectile was deduced, and the results obtained were as follows:

$$v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}(\varpi u)^{\frac{3}{8}}\left(\frac{\Delta}{pc}\right)^{\frac{1}{4}}\left[1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right], \tag{69}$$

$$P = K\left(\frac{fa}{\tau}\right)\Delta\frac{(p\varpi)^{\frac{1}{2}}}{c^2}. \tag{70}$$

Having a choice of units, the inch, foot, pound avoirdupois, and second will be used. The data will then be as follows for (69):

$v =$ the initial velocity in feet per second.
$c =$ the calibre of the gun in feet.
$u =$ the travel of the projectile in feet.
$p =$ the weight of the projectile in pounds.
$\varpi =$ the weight of the charge in pounds.
$\Delta =$ the density of loading (see definition, Chapter I).
$f =$ the force of the powder.
$\tau =$ the time in seconds of the combustion of a grain of the powder in free air.
$a, \lambda =$ two numerical coefficients depending upon the *form* of the grain.
$A, B =$ numerical constants, invariable for any gun and powder, independent of the elements of firing.

For (70) the data are given in the same units except that the calibre of the gun is given in inches. K is an invariable numerical constant.

73. Now, in order to be able to use the formulas we must know A, B, and K, and the quantities $\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}$ and $\frac{\lambda}{\tau}$, the latter quantities being variable for the particular powder used.

74. The Characteristics of the Powder.—The quantities $\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}$ and $\frac{\lambda}{\tau}$ are called the characteristics of the powder, because by their aid we are able to compare the results given by different powders in the same gun, with the same weights of charge and projectile.

Let $\left(\frac{fa}{\tau}\right)^{\frac{1}{2}} = \alpha$, and $\frac{\lambda}{\tau} = \beta$, and in this form they will hereafter appear in the formulas for velocity and pressure.

75. The quantity f is not easily determined practically, and the usual method is to adopt some powder as a standard and call its *force* unity. Then we may refer the force of all other powders to this value taken as the standard. It is convenient also to adopt unity as the value of τ for the standard powder. Then, if the powder is well defined as to form, the quantities a and λ, which depend alone on the form, may easily be determined, and the values of α and β for the standard powder are respectively equal to a and λ. a and λ are determined by formulae in which the quantities involved are the thickness of the grain, the ratio of the smallest to the other dimensions, and the number of grains in a unit of weight of the powder, as will hereafter be explained.

Sarrau adapts as a standard Wetteren powder [13–16], the shape being a flat grain with a square base. The number of firings in the United States with modern powders at this date (1887) being comparatively limited, Sarrau's standard will be accepted for the purpose of comparing U. S. powders with the French and with each other. The choice of a standard is immaterial to the practical working of the formulas, provided enough firings have been made under similar conditions to fix A, B, and K accurately. Sarrau's values are founded on the results of a large number of experiments and are believed to be reliable.

76. The Determination of A and B.—Since the characteristics of the standard powder are known, as are also the elements of firing, the only unknown quantities in the velocity formula are A and B. Therefore two equations will be needed for their determination, which are obtained by firing the standard powder in two dissimilar guns and observing the initial velocity by chronograph. It is necessary that the guns should not be similar, otherwise the resulting equations will be nearly identical.

Two dissimilar guns of the *same calibre* may sometimes be used, but only when there is a variation of the *weight of the projectile* and the *travel of the projectile*, and in any case the guns should be so selected as to give as large a variation of the quantity $\frac{(pu)^{\frac{1}{2}}}{c}$, which appears in the second term, as is possible; the greater the variation, the greater the accuracy of the determination of A and B. Writing α and β for $\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}$ and $\frac{\lambda}{\tau}$ respectively, we have

$$v = A\alpha\,(\varpi u)^{\frac{3}{8}}\left(\frac{\Delta}{pc}\right)^{\frac{1}{4}}\left[1 - B\beta\frac{(pu)^{\frac{1}{2}}}{c}\right].$$

Now in this formula put

$$\alpha\,(\varpi u)^{\frac{3}{8}}\left(\frac{\Delta}{pc}\right)^{\frac{1}{4}} = \frac{1}{X},$$

and

$$\beta\frac{(pu)^{\frac{1}{2}}}{c} = Y;$$

then, with the data obtained by firing the two guns, with the given conditions of loading, and the observed velocities v_1 and v_2, we have the following equations:

$$v_1 = \frac{A}{X_1}(1 - BY_1),$$

$$v_2 = \frac{A}{X_2}(1 - BY_2),$$

or

$$X_1v_1 = A - ABY_1,$$
$$X_2v_2 = A - ABY_2,$$

whence

$$AB = \frac{X_2v_2 - X_1v_1}{Y_1 - Y_2},$$

$$A = X_1v_1 + ABY_1 = \frac{Y_1X_2v_2 - Y_2X_1v_1}{Y_1 - Y_2}.$$

Having found A, the value of B easily follows.

77. The guns used by Sarrau for the numerical determination of A and B were a 19 cm. and a 10 cm., the elements of firing being as follows:

	c dm.	u dm.	p kg.	ϖ kg.	Δ	v dm.
19 cm. gun, . .	1.94	32.9	75	15	0.870	4480
10 cm. gun, . .	1.00	22.6	12	3.1	0.957	4850

The standard powder, for which $f = 1$, $\tau = 1$, has the following characteristics:

$$f = 1, \ \tau = 1, \ a = 2.572, \ \lambda = 0.851,$$
$$\alpha = (2.572)^{\frac{1}{2}}, \qquad \lambda = 0.851,$$
$$\log \alpha = 0.20513, \ \log \beta = 9.92993 - 10.$$

Substituting the above data in the formulas for X, Y, we find the values of A and B to be as follows, in our units, foot, pound, and second:

$$A = 502.63, \qquad B = .0058891,$$
$$\log A = 2.70124, \ \log B = 7.77005 - 10.$$

78. The formula for velocity then becomes for any powder in any gun,

$$v = 502.63\,\alpha\,(\varpi u)^{\frac{3}{8}} \left(\frac{\Delta}{pc}\right)^{\frac{1}{4}} \left[1 - .0058891\,\beta\,\frac{(pu)^{\frac{1}{2}}}{c}\right], \qquad (71)$$

the units being foot, pound avoirdupois, and second; that is to say, ϖ and p are expressed in pounds, u and c in feet, and v in feet per second. The values of A and B are invariable for all guns and all powders, but to use the formula we must find the values of α and β for the particular powder used, the method for which will hereafter be given.

79. On the Error committed in the determination of A and B.—It is easy to determine the effect of an error in the measurement of the velocities upon the determination of A and B.

Let ε be the absolute value of the greatest error which is to be apprehended in the measurement of the velocities v_1 and v_2. The error in AB and A is a maximum when the error affects v_2 and v_1 in opposite senses; we have consequently the two limits

$$\varepsilon_1 = \frac{X_2 + X_1}{Y_1 - Y_2}\,\varepsilon,$$
$$\varepsilon_2 = \frac{Y_1 X_2 + Y_2 X_1}{Y_1 - Y_2}.$$

The first concerns AB and the second A. Evidently, from these values $Y_1 - Y_2$ should be as large as possible by the choice of proper conditions of firing, and this accords with what has already been said on this point.

80. The Maximum Pressure on the Breech of the Gun.—The formula for maximum pressure, $P = K\frac{fa}{\tau}\,\Delta\,\frac{(p\varpi)^{\frac{1}{2}}}{c^2}$, represents,

as has already been stated, the maximum pressure on the base of the projectile. This value was obtained by a consideration of the maximum acceleration of the projectile, and the values of the exponents were thus deduced, but these values will not give correct results of the maximum pressure on the breech for variable weights of charge and projectile in the same gun or in other guns, except when the ratio of the weight of the charge to that of the projectile is the same, or nearly the same, as that used in determining the constant K. Therefore, in order that the formula should represent the pressure at any point on the walls of the powder chamber, it is necessary to assume that the pressure is uniform at any point in rear of the base of the projectile at the instant of maximum pressure. We know, however, that the pressure is not uniform at all points, and varies according to an unknown and complex law, probably.

We may, however, arrive at values of the exponents of ϖ and p which will represent correctly the pressure on the face of the breech plug of B. L. guns, or at the bottom of the bore of M. L. guns, where the pressures are usually measured by pressure gauges, as follows: Let us consider at any instant of motion a section S perpendicular to the axis of the gun and at some point between the bottom of the bore and the base of the projectile, and examine the system composed of the products of combustion between the plane S and the projectile.

According to a well known theory, the sum of the products of the masses of these portions of the charge in combustion by their acceleration along the axis of the bore is equal to the sum of the external forces projected on that axis. The external forces are the total pressure exerted on S and the reaction of the projectile on the powder gases.

If, then, we designate by

y the pressure per unit surface on the base of the projectile,
y_0 the pressure on the surface S,
ω the right section of the bore, supposed constant for all its length,
u the mass of the charge in front of S,
du any element of the mass u,
w the acceleration of the element du along the axis of the bore,

we have by the theory quoted,

$$(y_0 - y)\,\omega = \int w du. \qquad (a)$$

So that the difference between the pressure exerted on any layer and the pressure on the base of the projectile at any instant is

proportional to the sum of the products of the masses in front of the layer by their respective accelerations.

81. The integral $\int wdu$ is equal to the product of the mass u by a *mean* of all the accelerations w.

Denoting by θ the ratio of this mean acceleration to that of the projectile, its value of the latter being obtained by dividing the mass m of the projectile by the force $y\omega$ which produces the acceleration, we are able to write

$$\int wdu = \theta \frac{uy\omega}{m},$$

and consequently (a) becomes $y_0 = y\left(1 + \theta \frac{u}{m}\right)$.

The value of θ is unknown, and varies also, at each instant, with the position of the layer considered and also with all the variable elements of loading. We know, however, that it is positive; in fact, the velocities of nearly all the elements of the charge increase with the time, consequently it follows that the mean acceleration of these elements is positive; therefore the coefficient θ, which is proportional to this acceleration, is positive. We should then consider $\frac{y_0}{y}$ as an increasing function of the quantity $\frac{u}{m}$.

82. The general custom is to measure the maximum pressure at the breech, and if we suppose the layer S to coincide with the bottom of the bore, the preceding discussion leads us to admit that the ratio of the maximum pressure on the breech to that on the base of the projectile increases with the ratio $\frac{\varpi}{p}$ of the weight of the charge to that of the projectile; consequently the pressure on the breech should vary more rapidly with a change in the value of ϖ than a change in the value of p than the formula for maximum pressure on the base of the projectile indicates.

Let us denote by P_0 and P the maximum pressures on the breech and projectile respectively; then, from what precedes, we may write

$$P_0 = P\left(1 + \theta \frac{\varpi}{p}\right),$$

and attempt to represent the results of experience by giving a constant value to θ, or we may assume that the ratio $\frac{P_0}{P}$ varies as some as yet unknown power of $\frac{\varpi}{p}$,

or
$$\frac{P_0}{P} = \left(\frac{\varpi}{p}\right)^\gamma,$$

and
$$P_0 = K_0 \left(\frac{\varpi}{p}\right)^\gamma \frac{fa}{\tau} \Delta \frac{(p\varpi)^{\frac{1}{2}}}{c^2}.$$

Numerous experiments made by Sarrau show that the value of γ is sensibly $\frac{1}{4}$, and we have finally,

$$P_0 = K_0 \frac{fa}{\tau} \Delta \frac{\varpi^{\frac{3}{4}} p^{\frac{1}{4}}}{c^2}.$$

This formula will then be hereafter meant when speaking of the formula for maximum pressure on the gun. One firing determines K_0, and Sarrau finds with the standard powder that the value of K_0 is 17820, and log $K_0 = 4.25092$; hence, may be written

$$P_0 = 17820\, a^2 \Delta \frac{\varpi^{\frac{3}{4}} p^{\frac{1}{4}}}{c^2}, \qquad (72)$$

in which formula $a = 2.572$ for the powder adopted as the standard.

83. The Determination of α and β for any Powder.—An inspection of (71) shows that if a given powder be fired in two dissimilar guns and the initial velocities be observed, we have two resulting equations, A and B being known, from which α and β may be found by exactly the same method as was followed in the determination of A and B. This is the most accurate method, too, if the guns are well chosen. The object of obtaining values of α and β is to compare one powder with another, to compare the *force* and time of burning in free air, which may be easily deduced from the values of α and β, relative to some powder as a standard.

Whenever possible, then, α and β should be determined by firings in two dissimilar guns; but it is not always possible to obtain such data, and the following plan may be used to compare different powders:

The formula for velocity contains $\alpha = \left(\frac{fa}{\tau}\right)^{\frac{1}{2}} \beta = \frac{\lambda}{\tau}$, and the formula for maximum pressure on the breech contains $a^2 = \frac{fa}{\tau}$. Evidently, if we know the form of the grain, the number of grains to unit of weight, and the density, we may deduce α and λ from these data. From the pressure formula

$$P_0 = K_0 a^2 \Delta \frac{\varpi^{\frac{3}{4}} p^{\frac{1}{4}}}{c^2},$$

if P_0 is observed by crusher gauge, the value of a^2 may be found, since it is the only unknown quantity in the equation.

If at the same time the initial velocity is observed, α being found from the pressure formula, the value of β may be found, since it is then the only unknown quantity in the equation for velocity. These values of α and β thus found are relative to those of the standard powder. Since a and λ are known from the form and dimensions of the grain, we have

$$\tau = \frac{\lambda}{\beta}, \text{ and } f = \frac{a^2\tau}{a},$$

and the values of τ and f thus obtained are relative to the values of τ and f for the standard powder, which are assumed to be unity.

84. To find the Least Dimension of the Grain.—The least dimension of the grain is generally deduced from the density, and from the number of grains in a kilogram, the kilogram being selected as unit of weight because the comparison of volume and weight can best be made in French units.

85. Spherical Grains.—Let a spherical grain have a radius r and density δ, and let N be the number of grains in a kilogram. The weight of a grain is then $\frac{4}{3}\pi r^3\delta$,

$$\therefore\ \tfrac{4}{3}\pi r^3\delta N = 1,$$

whence

$$r = \left(\frac{3}{4\pi\delta N}\right)^{\frac{1}{3}}.$$

86. Grains of the Form of a Parallelopiped.—Let the three dimensions of the grain be b, m, and l; b being the smallest. Then put $\frac{b}{m} = x$, and $\frac{b}{l} = y$. We have then

$$bml\delta N = \frac{b^3}{xy}\,\delta N = 1,$$

whence

$$b = \left(\frac{xy}{\delta N}\right)^{\frac{1}{3}}.$$

When the grain has a square base, $x = y$, and $b = \left(\frac{x^2}{\delta N}\right)^{\frac{1}{3}}$;

$$\therefore\ x = (b^3\delta N)^{\frac{1}{2}}.$$

When the grain is cubical, $b = m = l$, and $b = \left(\frac{1}{\delta N}\right)^{\frac{1}{3}}$.

87. Pierced Cylindrical Grains.—Let R = external radius, r = internal radius, and h = height of cylinder.

Let $y = \frac{R}{r}$, and

$$x = \frac{R - r}{h} = \frac{R - \frac{R}{y}}{h},$$

$$\therefore\ h = \frac{yR - R}{xy},$$

$$\therefore\ \pi\left(\frac{y^2R^2 - R^2}{y^2}\right)\left(\frac{yR - R}{xy}\right)\delta N = 1,$$

$$\therefore\ R = \left[\frac{xy^3}{\pi(y^2 - 1)(y - 1)\delta N}\right]^{\frac{1}{3}},$$

whence the last dimension $= R - r$.

88. The value of a and λ for grains of different forms in service powders is as follows:

Sphero-Hexagonal,
Hexagonal,
Mammoth,
Cubical,
Cannon,
Rifle and all
irregular shaped grains, $\Big\}$ $a = 3,\ \lambda = 1,$

because irregular or regular polyedrons may be assumed to be inscribed or circumscribed by a sphere, and the law of burning will be that of a sphere approximately.

Pierced Prismatic Powder, $a = 1 + x,\ \lambda = \frac{x}{1 + x}$.—This powder is the form of the German Cocoa powders and Dupont's Brown powders, and they may be considered as burning approximately as pierced cylinders. In these powders $x = \frac{1}{2}$; hence for them,

$$a = \tfrac{3}{2},\ \lambda = \tfrac{1}{3}.$$

In the U. S. Army a square prismatic powder is sometimes used; hence the values of a, λ for such powders are

$$a = 1 + x + y;\ \lambda = \frac{x + y + xy}{1 + x + y};$$

but the base being square, $x = y$, and

$$a = 1 + x^2;\ \lambda = \frac{2x + x^2}{1 + x^2}.$$

For the Army Prismatic powder, $x = .720$; $\therefore$ for this powder

$$a = 2.442,$$
$$\lambda = .803.$$

These three forms will include all the service powders.

89. On the Use of the Formulas in Practice.—To sum up: by firing a given powder in two guns dissimilar in character, the constants $A_1 = A\alpha$, and $B_1 = \beta B$ are determined; if we wish to find the velocity this powder will give in any other gun, we write the formula as follows:

$$v = A_1 (\varpi u)^{\frac{3}{8}} \left(\frac{\Delta}{pc}\right)^{\frac{1}{4}} \left[1 - B_1 \frac{(pu)^{\frac{1}{2}}}{c}\right].$$

If we wish to compare a given powder with another, or with the standard, or if we have firings from but one gun, α and β are determined from the observed velocity and pressure. Since A and B are known, having been deduced from firings with the standard powder; with α and β and the known values of A and B we may evidently find the velocity the powder will give in any gun. It is, however, recommended to obtain A_1 and B_1 by two firings whenever possible.

EXAMPLES.

1. It is required to determine A_1 and B_1 by means of the records of the two firings with Dupont's black sphero-hexagonal powder of $\delta = 1.78$ given below, so that the resulting formula shall be applicable, without conversion of units, to quantities stated in feet and pounds. The data are, in feet and pounds:

VI-inch B. L. R. (S. B.).	60-pdr. B. L. R.
$\varpi = 32$	$\varpi = 10$
$\Delta = .9627$	$\Delta = .8987$
$u = 13.08$	$u = 7.67$
$p = 75.4$	$p = 46.5$
$c = 50$	$c = .44$
$v = 2001$	$v = 1427$

$$A_1 = 806.50.$$
$$B_1 = .00570.$$

2. It is required to find the muzzle velocity in the VI-inch (South Boston) gun when loaded with the powder defined in Example 1, as stated below in *pounds*, *feet*, and *seconds*.

$\varpi = 23$	$u = 13.08$
$\Delta = .692$	$c = .50$
$p = 67$	$v = 1748$ (measured).

By calculation, we find $v = 1742$.

3. It is required to find the muzzle velocity in the 8-inch M. L. R., when loaded with the powder defined in Example 1, as stated below in *pounds, feet,* and *seconds.*

$\varpi = 35$	$u = 8.63$
$\Delta = .9226$	$c = .667$
$p = 180$	$v = 1377$ (measured).

By calculation, we find $v = 1355$.

4. Find the values of A_1 and B_1 for Schaghticoke navy rifle powder from the following data, in feet and pounds, in the 60-pounder B. L. R. and 3-inch B. L. R.

60-pdr. B. L. R.	3-inch B. L. R.
$\varpi = 6$	$\varpi = \frac{3}{4}$
$\Delta = .5392$	$\Delta = .76$
$u = 7.67$	$u = 3.22$
$p = 46.5$	$p = 7$
$c = .44$	$c = .25$
$v = 1071$	$v = 983.$

Note.—The velocity here given for the 60-pounder is the mean of five determinations obtained at the Naval Ordnance Proving Grounds with the service charge of 6 pounds of rifle powder and a 46.5 pounds projectile (with expanding band); that given for the 3-inch is the mean of two determinations obtained at the Naval Academy with 12 ounces of rifle powder and the service projectile of 7 pounds weight.

Evidently the values of f, τ for this powder may be found.

5. It is required to find A_1 and B_1 for the use of Dupont's Brown Prismatic O. P. powder, $\delta = 1.818$, in any gun. The following data were given by two firings:

South Boston Gun.	8-inch B. L. R. mark I.
$\varpi = 25$ pounds.	$\varpi = 110$ pounds.
$p = 68$ pounds.	$p = 250$ pounds.
$u = 12.5$ feet.	$u = 16.41$ feet.
$C = 920$ cu. in.	$C = 3824$ cu. in.
$c = .5$ ft.	$c = .6667$ ft.
$v = 1716$ f. s.	$v = 1949$ f. s.
$P = 7.9$ tons.	$P = 15.6$ tons.

Ans. $A_1 = 625.00.$
$B_1 = .002998.$

6. How many grains weigh a pound in the case of pierced hexagonal prismatic grains, density 1.818, the radius of the cylindrical hole being $\frac{1}{4}$ inch, the distance between faces $1\frac{1}{2}$ inches, and the height of the grain 1 inch? *Ans.* 8.6938.

7. Equation (71), p. 82, may be written:

$$v = A\,(\varpi u)^{\frac{3}{8}} \left(\frac{\Delta}{pc}\right)^{\frac{1}{4}} \left[1 - B\,\frac{(pu)^{\frac{1}{3}}}{c}\right].$$

It is desired from the data below, to determine A and B so that the resulting equation shall be applicable, without conversion of units, to quantities stated in *feet*, *pounds*, and *seconds*. The data are, in the units just stated:

$\varpi = 29.125$	$\varpi = 50$
$u = 10.0$	$u = 11.62$
$\Delta = .8763$	$\Delta = .9886$
$p = 51$	$p = 100$
$c = .50$	$c = .50$
$v = 1685$	$v = 1836$

We find

$$A = 497.6.$$
$$B = .0014004.$$

Note.—The above records are from the two VI-inch B. L. R.'s known as the South Boston gun (on the left) and Dolphin's gun (on the right); the former had its brass muzzle-piece off. The powder was German cocoa (C_{82}), prismatic grains of $\delta = 1.867$ and 10.06 granules per pound. The granules are hexagonal in section, about 1 inch in height, and the difference of the radii of the axial hole (of which there is one) and of the circle inscribed in the hexagon is about $\frac{1}{2}$ inch.

8. If we determine α and β referred to standard powder from the data given in Example 1, we shall be able to find the force and time of burning of the grains of powder there defined. We find

$$\alpha = 0.99001.$$
$$\beta = 0.2378.$$

From these we find $f = 0.9158$, $\tau = 1.402$, $v = 0.8$ cent. per second.

9. The measured pressure in the South Boston gun in the firing recorded in Example 1 was 12,100 pounds per square inch; from the data given in Example 1 determine K_1, by means of which the pressure can be computed when this powder is used in any other gun.

Ans. $K_1 = 14836$.

10. Find, from the value of K_1 just determined, the pressure in the Dolphin's gun when loaded as defined in Example 1 with cocoa powder. By calculation, we find $P = 24230$.

11. It is required to compare calculated with measured pressures in the following record of firings in the VI-inch B. L. R. (Dolphin's gun, chamber capacity $= 1400$ cubic inches), with black sphero-hexagonal powder of 60 granules to the pound, and $\delta = 1.817$, and $p = 100$ pounds, all the projectiles being identical in every respect.

$\varpi = 35$ lbs.	$P = 16900$ lbs. per square inch	$V = 1506$ f. s.
$\varpi = 40$ "	$P = 22350$ "	$V = 1657$ f. s.
$\varpi = 45$ "	$P = 26650$ "	$V = 1760$ f. s.
$\varpi = 47$ "	$P = 27850$ "	$V = 1792$ f. s.
$\varpi = 49$ "	$P = 36700$ "	$V = 1880$ f. s.

For convenience, the formula may be written

$$P = k_1 \varpi^{\frac{7}{4}};$$

whence, from the first record, we find $k_1 = 33\ 55$. The other calculated pressures are therefore 21350, 26240, 28310, 30450.

12. Find the value of a^2 from the data of Example 9, and compare with the result in 7. *Ans.* $a = 1.0155$.

13. Derive, from the values of A_1 and B_1 found in Example 6 for the cocoa powder there defined, the formula giving the velocity at any point in the 8-inch hooped B. L. R. when loaded as follows with this powder, in *pounds*, *feet*, and *seconds*.

$$\varpi = 125 \qquad \Delta = .9049$$
$$p = 250 \qquad c = .6667$$

$$v = 825.85 u^{\frac{3}{8}} - 27.428 u^{\frac{7}{8}}.$$

Putting here $u = 16.41$, we find $v = 2041$ f. s. for the muzzle velocity.

14. Show from the equation in Example 12 that the greatest attainable velocity in the 8-inch gun when loaded as there defined is 3213 f. s., and that the length of the shot's travel must be 166.5 feet to reach it.

15. If in a certain gun 54 pounds of powder gave 2105 f. s. velocity with 15.6 tons pressure, what velocity and pressure will 52 pounds give with the same gun and powder? *Ans.* $V = 2056$ f. s.
$P = 14.7$ tons.

16. Find the weight of charge necessary to obtain 1700 f. s. velocity

in the 6-inch B. L. R. if 54 pounds of the same powder gave 2105 f. s. with 15.6 pressure. What will be the pressure?

Ans. $\varpi = 38.36$ lbs.
$P = 9.34$ tons.

17. Suppose the projectile is not pushed home to its seat, and if the variation of travel, du, is known, deduce a formula showing its effect on the velocity and pressure.

Ans.

$$v = Cu^{\frac{3}{8}} \Delta^{\frac{1}{4}} (1 - Du^{\frac{1}{2}}),$$

$$\therefore \frac{dv}{v} = \frac{3}{8} \frac{du}{u} + \frac{1}{4} \frac{d\Delta}{\Delta} - \frac{1}{2} \frac{Ddu}{(1 - Du^{\frac{1}{2}}) u^{\frac{1}{2}}},$$

$$dP = \frac{d\Delta}{\Delta} P.$$

18. The formula for velocity in the 6-inch B. L. R. as a function of u, in the case of O. P. powder, $\varpi = 54$ pounds, $p = 100$ pounds, is

$$v = 1056.8u^{\frac{3}{8}} - 64.409u^{\frac{7}{8}},$$

what travel of the projectile will give a maximum velocity, and what is the maximum velocity attainable in a 6-inch gun with the above weight of charge, powder, and projectile? *Ans.* $u = 49.4$ feet.
$v = 2607.7$ f. s.

The following tables may be used to furnish numerous examples for practice in the use of the formulas, and, for convenience, the latter are here repeated:

$$V = Aa (\varpi u)^{\frac{3}{8}} \left(\frac{\Delta}{pc}\right)^{\frac{1}{4}} \left[1 - B\beta \frac{(pu)^{\frac{1}{2}}}{c}\right],$$

$$A = 502.63, \qquad B = 0.0058891,$$

$$\log A = 2.70125, \ \log B = 7.77005 - 10,$$

$$\left.\begin{aligned} P_0 &= K_0 a^2 \Delta \frac{\varpi^{\frac{3}{4}} p^{\frac{1}{4}}}{c^2}, \\ K_0 &= 17820, \\ \log K_0 &= 4.25091, \end{aligned}\right\}$$

Note.—In this formula c is given in inches.

$$\tau = \frac{\lambda}{\beta}; \ f = \frac{a^2 \tau}{a}.$$

It must be remembered that the values of τ and f are relative to the corresponding values for standard powder, which are assumed to be unity. The value of τ for the standard powder is approximately equal to the time of burning in free air of a grain of that powder, on the supposition that the velocity of combustion is 0.1 decimetre a second. For reference, the characteristics of the standard powder are repeated in the tables.

Table I.

Powder.	Guns.	δ	a	λ	C feet.	P pounds.	ϖ pounds.	u feet.	Δ	v feet.	P_0 pounds.	Remarks.
Standard..................		1.75	2.572	0.851								
LX Dupont..............	3.2″ U. S. A.	1.706	2.442	0.803	0.265	13	3.5	6.1	0.857	1649	31000	
LXB "	3.2″ U. S. A.	1.706	2.442	0.803	0.265	13	3.75	6.1	0.827*	1756	35150	Freyer Gas Check.*
IKD "	3.2″ U. S. A.	1.725	3.0	1.0	0.265	13	3.5	6.1	0.857	1630	29100	
IB 123....................	3.17″ U. S. A.	1.728	3.0	1.0	0.264	10.5	5.45	6.2	0.814	1933	25000	
NM Dupont..............	12″ B. L., U. S. A.	1.833	1.5	0.333	1.00	800	265	22.7	0.827	1688	26350	
NV_3 "	12″ B. L., U. S. A.	1.826	1.5	0.333	1.00	800	265	22.7	0.827	1718	26890	
NR "	12″ B. L., U. S. A.	1.814	1.5	0.333	1.00	800	265	22.7	0.827	1826	32990	
NV_1 "	12″ B. L., U. S. A.	1.830	1.5	0.333	1.00	800	265	22.7	0.827	1760	26625	
NV_2 "	12″ B. L., U. S. A.	1.818	1.5	0.333	1.00	800	265	22.7	0.827	1756	28000	
Sp. Hex. 98.......	6″ B. L., U. S. N.	1.800	3.0	1.00	0.5	100	40	11.62	0.792	1816	29600	Mark I.
Sp. Hex. 100............	6″ B. L., U. S. N.	1.800	3.0	1.00	0.5	100	47	11.62	0.931	1978	35840	"
German Cocoa..........	6″ B. L., U. S. N.	1.846	1.5	0.333	0.5	100	40	11.62	0.792	1717	23520	"
German Cocoa..........	6″ B. L., U. S. N.	1.846	1.5	0.333	0.5	100	47	11.62	0.931	1925	30900	"
Sp. Hex. 100............	6″ B. L., U. S. N.	1.830	3.0	1.00	0.5	100	47	11.62	0.931	1906	42560	"
NU Dupont..............	6″ B. L., U. S. N.	1.818	1.5	0.333	0.5	100	50	12.21	0.972	1881	27300	Mark II.
OP "	6″ B. L., U. S. N.	1.818	1.5	0.333	0.5	100	50	12.21	0.972	1983	30464	"
OP "	8″ B. L., U. S. N.	1.818	1.5	0.333	0.6	250	110	16.41	0.798	1949	35000	Mark I.

With the data in Table I, the formulas gave the following results for α, β, f, and τ.

TABLE II.

Powder.	α	β	f	τ	Remarks.
Standard	1.6036	0.851	1.0	1.0	Wetteren (13-16) Flat grain.
LX Dupont	2.0683	1.452	0.9689	0.553	Square Prism, N = 270.
LXB "	2.1847	1.4838	1.0578	0.54119	" " N = 270.
IKD "	2.0039	1.3791	0.9706	0.7251	Granulated, N = 2200.
IB "	1.6424	0.83524	1.0766	1.1973	Sp. Hexagonal, N = 123.
NM "	0.8585	0.2083	0.7231	1.5987	Pierced Prismatic, N = 11½.
NV_3 "	0.8673	0.2004	0.8332	1.6616	" " " "
NR "	0.9606	0.2432	0.8422	1.369	" " " "
NV_1 "	0.8630	0.1691	0.9777	1.9689	" " " "
NV_2 "	0.8850	0.1987	0.8752	1.6762	" " " "
Sp. Hex. 98	1.2252	0.4214	1.1924	2.3753	Sp. Hexagonal, N = 98.
Sp. Hex. 100	1.2762	0.3497	1.2762	2.7944	" " N = 100.
German Cocoa	1.0922	0.2958	0.8950	1.1256	Pierced Prismatic, N = 11½.
German Cocoa	1.0856	0.2527	1.0353	1.3177	" " " "
LN Dupont	1.2755	0.6047	0.8968	1.6538	Sp. Hexagonal, N = 100.
NU "	0.9768	0.1804	1.1770	1.846	Pierced Prismatic, N = 11½.
OP "	1.0319	0.1848	1.2787	1.8014	" " " "
OP "	1.0800	0.3191	0.8114	1.0435	" " " "

CHAPTER IX.

THE INFLUENCE OF THE CHARACTERISTICS OF THE POWDER ON VELOCITY AND PRESSURE.

90. The formula for maximum pressure on the bottom of the bore (72) shows that in the same gun and with the same conditions of loading the pressure is proportional to $\frac{fa}{\tau}$.

f is the force of the powder, and depends principally upon the proportions of the ingredients and their character.

a depends on the form of the grain.

τ varies with the nature of the powder, its thickness, and the method of fabrication.

If these three elements vary in such a manner that $\frac{fa}{\tau}$ remains constant, the maximum pressure will remain constant.

As to the velocity under the given conditions, it depends upon $\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}$ and $\frac{\lambda}{\tau}$; λ is also a quantity which depends on the form of the grain.

The formula for velocity (69) is composed of two terms, of which the first term represents the ideal case in which the form of the grain is such that the velocity of emission of gas in free air is uniform, and in which case the expression $\psi(t)$ reduces to its first term. The second term is subtractive, and represents the effect of the decrease of the velocity of emission of gas in free air when the grains are in the forms which are admissible in practice; it is evidently an advantage to make the second term as small as possible.

91. Take two powders having the characteristics

$$f, \quad a, \quad \lambda, \quad \tau,$$
$$f', \quad a', \quad \lambda', \quad \tau'.$$

If the pressure remains constant,

$$\frac{fa}{\tau} = \frac{f'a'}{\tau'};$$

and if this be true, the first terms of the formulas for velocity are also constant, but the subtractive terms have the following ratio:

$$\frac{\lambda}{\tau} : \frac{\lambda'}{\tau'} \quad \therefore \quad \frac{\lambda'\tau}{\lambda\tau'};$$

and if this ratio is less than unity, the powder whose characteristics are λ', τ' will give the *higher velocity* with the *same maximum pressure.*

92. The Influence of the Force of the Powder.—Suppose the form of the grain is the same in the two powders, or $a = a'$, and we have from the equality of pressures $\frac{f}{\tau} = \frac{f'}{\tau'}$, and the subtractive terms are then to each other as

$$\frac{1}{\tau} : \frac{1}{\tau'}.$$

This ratio is equal to $\frac{f}{f'}$.

Consequently, for the same form of grain the stronger powder will give the *higher velocity* with the *same maximum pressure*, the times of combustion being proportional to the force of the powder.

93. Influence of the Form and Dimensions of the Grain.— Consider, in the same gun, two powders which are equally strong, but which differ in the form and dimensions of the grain. Putting $f = f'$, the condition of equality of pressures becomes, see (72),

$$\frac{a}{\tau} = \frac{a'}{\tau'},$$

and the ratio of the subtractive terms of the velocity is, see (69),

$$\frac{\lambda' a}{\lambda a'}. \tag{73}$$

For the spherical or cubical grain, we have (Chap. VII),

$$a = 3, \ \lambda = 1.$$

Let us compare this grain with other possible forms.

1*st. Cylindrical pierced grain.*—This grain has two dimensions: height and thickness. Calling x the ratio of the least to the greatest, we have (Chap. VII),

$$a' = 1 + x, \ \lambda' = \frac{x}{1 + x}.$$

The ratio has then the value

$$\frac{3x}{(1 + x)^2}. \tag{74}$$

This function of x is a *maximum* for $x = 1$: its value then being $\frac{3}{4}$. Consequently, by using *cylindrical pierced grains of equal height and thickness*, the subtractive term corresponding to spherical or cubical grains may be diminished by *one-fourth* of its value. If its absolute value is 300 feet, for example, we would gain 75 feet of velocity without increase of maximum pressure. The gain increases for decreasing values of x; if we take $x = \frac{1}{2}$, the subtractive term is diminished by *one-third;* and, supposing the same absolute value of the velocity, the increase is about 100 feet.

2d. Grains of the form of a parallelopiped.—Calling x and y the ratio of the least side to the two others, we have (No. 42),

$$a = 1 + x + y,\ \lambda = \frac{x + y + xy}{1 + x + y}, \tag{75}$$

and the ratio of the subtractive term to that for the spherical or cubical grain is

$$\frac{3(x + y + xy)}{(1 + x + y)^2}. \tag{76}$$

If the grain has a square base, we have $y = x$; and the above ratio becomes

$$\frac{3(2x + x^2)}{(1 + 2x)^2}. \tag{77}$$

This expression diminishes as x varies from 1 to 0, but the change is somewhat slow. When two dimensions of the grain are double the third, $x = \frac{1}{2}$, and the value of the ratio is $\frac{15}{16}$. The velocity therefore increases by $\frac{1}{16}$ of the subtractive term corresponding to the spherical grain. Supposing this term 300 feet, it is about 21; it becomes 48 for $x = \frac{1}{3}$.

The advantage of flattened grains over those which are spherical or cubical seems, then, real; but the advantage appears to be less than in the case of grains which are pierced.

94. The Influence of the Time of Combustion of the Grain in Free Air. The Theoretical Maximum of Velocity.

—If now we neglect the condition that the maximum pressure shall remain constant, we may find the influence of the time of combustion, τ, on the initial velocity for any powder of determined form and nature.

Formula (69) is

$$v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}(\varpi u)^{\frac{3}{8}}\left(\frac{\Delta}{pc}\right)^{\frac{1}{4}}\left[1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right];$$

and if we vary τ, other quantities remaining constant, the value of v will pass through a maximum.

If we put for brevity,

$$\varphi(\tau) = \tau^{-\frac{1}{2}} - B\,\frac{\lambda(pu)^{\frac{1}{2}}}{c}\,\tau^{-\frac{3}{2}}, \tag{78}$$

(69) becomes

$$v = A(fa)^{\frac{1}{2}}(\varpi u)^{\frac{3}{8}}\left(\frac{\Delta}{pc}\right)^{\frac{1}{4}}\varphi(\tau). \tag{79}$$

The value of τ corresponding to the maximum of φ, and therefore of v, is given by the condition $\varphi'(\tau) = 0$, that is to say, by the relation

$$B\,\frac{\lambda(pu)^{\frac{1}{2}}}{\tau c} = \frac{1}{3}. \tag{80}$$

Denoting the particular value of τ by τ_1, we have

$$\tau_1 = 3B\,\frac{\lambda(pu)^{\frac{1}{2}}}{c}. \tag{81}$$

The corresponding value of v is found from the relation

$$v_1 = \frac{2}{3}\left(\frac{fa}{\tau_1}\right)^{\frac{1}{2}}(\varpi u)^{\frac{3}{8}}\left(\frac{\Delta}{pc}\right)^{\frac{1}{4}}. \tag{82}$$

Replacing τ_1 by its value from (81), and v_1 becomes

$$v_1 = \frac{2}{3}A\left(3B^{-\frac{1}{2}}\right)\left(\frac{fa}{\lambda}\right)^{\frac{1}{2}}\frac{\varpi^{\frac{3}{8}}\Delta^{\frac{1}{4}}c^{\frac{1}{4}}u^{\frac{1}{8}}}{p^{\frac{1}{2}}}. \tag{83}$$

95. Ballistic Coefficient.—It results from formula (83) that for given values of the various elements of firing, the greatest velocity that a powder can produce is proportional to $\left(\frac{fa}{\lambda}\right)^{\frac{1}{2}}$. This factor, which has been called the *ballistic coefficient*, depends at the same time on the force of the powder and the form of the grain.

If the force is constant, the ballistic coefficient is proportional to $\left(\frac{a}{\lambda}\right)^{\frac{1}{2}}$. It depends only on the form of the grain.

When the grain is spherical or cubical we have $a = 3$ and $\lambda = 1$. If, then, we designate by C the ballistic coefficient of a powder referred to that of a spherical or cubical powder of the same force, we can put

$$C = \left(\frac{a}{3\lambda}\right)^{\frac{1}{2}}.$$

The value of C is greater than 1 for grains of the form of a parallelopipedon or a pierced cylinder. We can, then, by the employment of these grains, increase the velocity without changing the other elements of firing. But in practice this increase cannot pass a certain limit on account of the corresponding increase of the maximum pressure, which varies as the square of the ballistic coefficient, and rapidly approaches the limit imposed by the resistance of the piece.

Several experimental facts, notably those of Maguin, demonstrate that there exists for each gun and in each condition of firing, a size of grain, and consequently a duration of combustion of the grain, which gives a maximum of velocity.

It would be an advantage to adopt a maximum powder, not only because the charge would thus be better utilized, but also for the reason that the chance variations which are made in the manufacture of the powder, affecting the velocity of combustion, would not have a sensible influence on the initial velocity, which assures regularity of fire.

It should be remarked, however, that the employment of a maximum powder gives generally very high interior pressures, and this inconvenience is also more serious since irregularities in the fabrication can in this case produce great variations of pressure without sensibly modifying the velocity. This particularity is found from the special form of the functions which represent the velocity and the maximum pressure. The velocity is the difference of two terms increasing together as τ diminishes, so that their variations compensate near the maximum. The pressure is represented, however, by a monomial formula, and it varies rapidly in the inverse ratio of the period of combustion.

96. Calculation of the Period of Combustion.—In order to calculate the absolute value of τ which enters into these formulæ, it is necessary to know

1st. The velocity of combustion under the atmospheric pressure;

2d. The least dimension of the grain.

The first of these elements has been determined experimentally by Piobert for powders of different composition and fabrication. The result of his experiments is that, all other things being equal, the velocity of combustion of a powder *varies inversely with its density*. We have, therefore,

$$v_0 = \frac{c}{\delta}, \tag{84}$$

δ being the density of the grain, and c a constant depending upon the composition and degree of dryness of the material. It depends also upon the process of fabrication and on the period or duration of trituration, but the variations resulting would seem to be relatively small. We can with a sufficiently general approximation suppose that, within the limits of dryness realized in the habitual conditions of fabrication, the value of c to be nearly:

0.200 for powders for war purposes and for English powders;

0.130 for sporting powder (the units being the decimetre and the second).

With regard to the least dimension of the grain, it is deduced, generally, *from the density* and from the *number of grains in a kilogram.*

Influence of the various Elements of Firing on Velocity and Pressure.

97. In a cannon we may consider the quantities

$$c, u, p,$$

(calibre, length of path, and weight of projectile) as constants or given quantities. The variables with which, in connection with the constant quantities, we wish to obtain a given velocity, are

$$f, a, \lambda, \tau, \varpi, \Delta,$$

and there may be an infinite number of combinations made with these variables, giving the same velocity with different maximum pressures or the same pressure with different velocities.

It has been shown how we can, with fixed values of ϖ and Δ, lower the pressure and retain the velocity, or increase the velocity and retain the same pressure, by a proper choice of the variables

$$f, a, \lambda, \tau,$$

which refer exclusively to the powder.

98. We may consider as known also f, a, λ, that is to say, the force of the powder and the form of the grain, and evaluate the influence of ϖ, Δ, and τ on the values of the velocity and maximum pressure. With this in view, consider formula (79) for velocities. The factor $\varphi(\tau)$ we may write, taking into account equation (81), in an abbreviated form, thus:

$$\varphi(\tau) = \frac{3\tau - \tau_1}{3\tau^{\frac{3}{2}}}, \tag{85}$$

τ_1 being the value of τ corresponding to the maximum velocity with the form of grain adopted.

If, then, we suppose the quantities ϖ, Δ, and τ to be the only variables, we may put

$$v = K\varpi^{\frac{3}{8}}\Delta^{\frac{1}{4}}\tau^{-\frac{3}{2}}(3\tau - \tau_1), \tag{86}$$

K denoting a constant. The corresponding value of the maximum pressure is, according to equation (72),

$$P_0 = K_0 \frac{\varpi^{\frac{3}{4}}\Delta}{\tau}.$$

We wish to find the variation of v and P for very small increments, $d\varpi$, $d\Delta$, $d\tau$, of the variables. We find very easily:

$$\frac{dv}{v} = \frac{3}{8}\frac{d\varpi}{\varpi} + \frac{1}{4}\frac{d\Delta}{\Delta} - \frac{3}{2}\frac{\tau - \tau_1}{3\tau - \tau_1} \cdot \frac{d\tau}{\tau}, \tag{87}$$

$$\frac{dP_0}{P_0} = \frac{3}{4} \cdot \frac{d\varpi}{\varpi} + \frac{d\Delta}{\Delta} - \frac{d\tau}{\tau}. \tag{88}$$

The influence of each variable on the value of the velocity is measured by the coefficient which multiplies the relative variation of each variable in expression (40).

The coefficients of $\frac{d\varpi}{\varpi}$, $\frac{d\Delta}{\Delta}$, and $\frac{d\tau}{\tau}$ are respectively

$$\frac{3}{8}, \frac{1}{4}, \frac{3}{2} \cdot \frac{\tau - \tau_1}{3\tau - \tau_1}.$$

The third coefficient varies with τ and is equal to zero for $\tau = \tau_1$.

Its value increases with τ, but does not exceed $\frac{1}{4}$ except where τ is greater than $\frac{5}{3}\tau_1$, which is not likely to happen in the habitual conditions of practice.

98. It follows from the preceding that the three coefficients are arranged in the order of their relative value, and that the variations of each of the variables

$$\varpi, \Delta, \tau$$

have more influence than the one next on its right as arranged above.

On the contrary, the influence of ϖ on the maximum pressure is less than that of Δ or τ.

Let us suppose four cases.

99. 1. **ϖ constant, and Δ and τ variable.**—If the weight of charge is constant, and the density of loading and time of combustion are assumed to vary, the density of loading being made to vary

by an increase or decrease in the size of the powder chamber, by an examination of the formulas for velocity and pressure it is evident that in order to maintain the same pressure the time of burning must change relative to the change in the density of loading, and in such a way as to compensate for that change. For example, we may obtain greater and greater velocities by an increase of the density of loading, with the same charge, by a proper increase of the time of burning.

Hence the conclusion, in order to obtain the greatest velocities we should use high densities of loading and slow powder.

2. **Suppose Δ constant and ϖ, τ variables.**—Reasoning as in case 1, an increase in the weight of charge may be compensated for by a proper increase of τ, having due regard to the pressure, and an increase of velocity will result with the same travel of the projectile.

3. **τ constant, ϖ, Δ variables.**—If we fix the size of the grain, and consequently the time of combustion of the grain, τ; in order to increase the velocity we must increase the charge, and reduce the density of loading by increasing the powder chamber.

4. **Suppose ϖ, τ variables in a powder chamber of constant capacity.**—We have supposed in the three preceding cases that the size of the powder chamber can be increased or decreased at will, and in designing guns to perform certain work the conclusions reached are useful. Take the case, however, of a gun already built and which has a powder chamber of fixed capacity. The discussion of formulas (87) and (88) lead to the conclusion that it is generally of advantage to fill the powder chamber to its full capacity, provided that as we increase the charge we use a slower powder.

If we call S the capacity of the chamber, whence $\Delta = \frac{\varpi}{S}$, and if $n =$ the exponent of τ, $\frac{d\Delta}{\Delta} = \frac{d\varpi}{\varpi}$, $\therefore$ (87) and (88) become

$$\frac{dv}{v} = \frac{5}{8}\frac{d\varpi}{\varpi} - n\frac{d\tau}{\tau},$$

$$\frac{dP_0}{P_0} = \frac{7}{4}\frac{d\varpi}{\varpi} - \frac{d\tau}{\tau}.$$

94. If we suppose ϖ and τ to vary in such a manner that the maximum pressure remains constant,

$$\frac{d\tau}{\tau} = \frac{7}{4}\frac{d\varpi}{\varpi},$$

whence the relation

$$\frac{dv}{v}=\frac{7}{4}\left(\frac{5}{14}-n\right)\frac{d\varpi}{\varpi}.$$

Since n is sensibly less than $\frac{5}{14}$ in the usual conditions of practice, this relation shows that v increases with ϖ.

Examples.

1. It is required to find the correct granulation in the 6-inch B. L. R. when loaded with spherical powder of $\delta = 1.79$, and velocity of burning of .1411 decim. per second, and $p = 75.4$. In the gun considered (South Boston), $u = 157$ inches.

We find, using (81), and data from Example 6, Chap. VIII,

$$\tau_1 = .5779.$$
$$N = 112 \text{ per pound.}$$

2. Prove that, in similar and similarly loaded guns, the powder granules being all of the same form and having the same velocity of burning, the dimensions of the grain should vary directly as the calibre to make the velocities a maximum.

3. The radius of the grain defined by the conditions of Example 1 is 0.3207 inch. For powder of the same form, all guns in the Navy being assumed similar and similarly loaded; find in guns of what calibres the powders (assumed to be of spherical form) described in the table on page 338, Ordnance Instructions, should be used to get maximum velocities.

4. Find how many granules per pound will give the maximum velocity in the 3-inch B. L. R. when loaded with $p = 7$ lbs. For this gun we have $u = 38.61$ inches; whence, taking the granules, from their irregular form, to be spheres, and taking $\delta = 1.79$ and $v = .1411$ decim. per second, we find $\tau_1 = 0.175$

$$N = 4041 \text{ per pound.}$$

5. Taking, in Example 4, $v = .10$ decim. and $\delta = 1.75$; find the value of N. $N = 11544.$

6. The velocity and pressure in a certain gun with 30 pounds of powder are 2000 f.s. and 33,600 lbs. per square inch; find the approximate value of the velocity and pressure in case the charge weighs $30\frac{1}{4}$ lbs. $V' = 2010.$

$$P = 34020.$$

7. Show that the powder which gives the maximum velocity is

best with respect to a gun's accuracy as depending upon small variations in quantity of powder. Equation (40) shows this.

8. The velocity of recoil depends largely upon the momentum of the projectile; show that the latter, for changes of p, passes through a maximum when

$$p^{\frac{1}{2}} = \frac{3}{5} \cdot \frac{\tau c}{B\lambda u^{\frac{1}{4}}}.$$

9. The maximum powder is being used in a certain gun, the reduced length of whose powder chamber is 10 units. The whole length of the bore, including the chamber, is 100 units. Find the approximate value of the effect on the velocity and pressure due to the accidental pushing the base of the shot 1 unit too far into the bore from the breech. All elements not affected by the change stated will remain the same.

$$\frac{V}{V'} = 1.03.$$

$$\frac{P}{P'} = 1.11.$$

10. What has probably been the reasoning justifying the battering and ordinary charges laid down for the XI and XV-inch S. B. guns on p. 408, Ordnance Instructions? What would be about the ratio of the pressures with these two charges?

CHAPTER X.

MONOMIAL FORMULA FOR VELOCITY.

95. **The Modulus of the Powder.**—In the preceding chapter it has been established that of all the values of τ which may be substituted in the formula for velocity, there is one for which the velocity is a maximum. This value of τ we have called τ_1. We often speak of a powder, in practice, as a quick powder or a slow powder, but these definitions are not definite, and are often misleading. *In a given gun* a powder is slow when the time of its combustion is notably longer than that which in the particular gun corresponds to the theoretical maximum velocity; moreover, two powders fired in different guns should be considered as equivalent in quickness if their times of combustion are proportional to the respective values of τ which will give maximum velocities in the respective guns. Therefore, define the ratio of the time of burning of any given powder in a given gun to the time of burning that will give a maximum velocity in that gun as the *modulus* of the powder or a measure of its quickness. This ratio is usually written

$$x = \frac{\tau_1}{\tau},$$

in which x is the *modulus* of the powder. Sarrau adopts the following scale:

Value of the Modulus.	Qualification of the Powder.
1.0	Very quick.
0.9	Quick.
0.8	Medium.
0.7	Slow.
0.6	Very slow.

We have seen that

$$\tau_1 = 3B \frac{\lambda (pu)^{\frac{1}{2}}}{c},$$

and if $x = \dfrac{\tau_1}{\tau}$, $$x = 3B \frac{\lambda}{\tau} \frac{(pu)^{\frac{1}{2}}}{c}. \qquad (89)$$

96. The Velocity as a Function of the Modulus.—We can introduce x in place of τ in the formula for velocity and we obtain a new formula which is useful in calculations.

Recollecting the value of τ_1, (69) may be written,

$$v = A\left(\frac{fa}{\tau_1}\right)^{\frac{1}{2}} (\varpi u)^{\frac{3}{8}} \left(\frac{\Delta}{pc}\right)^{\frac{1}{4}} \left(\frac{\tau_1}{\tau}\right)^{\frac{1}{2}} \left(1 - \frac{1}{3}\frac{\tau_1}{\tau}\right). \tag{90}$$

Or better, putting $x = \frac{\tau_1}{\tau}$,

$$v = \frac{1}{3} A\left(\frac{fa}{\tau_1}\right)^{\frac{1}{2}} (\varpi u)^{\frac{3}{8}} \left(\frac{\Delta}{pc}\right)^{\frac{1}{4}} x^{\frac{1}{2}} (3 - x). \tag{91}$$

Now, replacing τ_1 by its value, and putting $f(x) = \frac{1}{2} x^{\frac{1}{2}} (3 - x)$, (91) becomes

$$v = \frac{2}{3} A (3B)^{-\frac{1}{2}} \left(\frac{fa}{\lambda}\right)^{\frac{1}{2}} \frac{\varpi^{\frac{3}{8}} \Delta^{\frac{1}{4}} c^{\frac{1}{4}} u^{\frac{1}{8}}}{p^{\frac{1}{2}}} f(x). \tag{92}$$

97. The Pressure as a Function of the Modulus.—The maximum pressure on the base of the projectile is

$$P = \frac{fa}{\tau_1} \frac{\Delta (p\varpi)^{\frac{1}{2}}}{c^2} \frac{\tau_1}{\tau}.$$

Putting $\frac{\tau_1}{\tau} = x$, and replacing τ_1 by its value in the denominator, we have

$$P = K(3B)^{-1} \left(\frac{fa}{\lambda}\right) \frac{\Delta \varpi^{\frac{1}{2}}}{cu^{\frac{1}{2}}} x. \tag{93}$$

The maximum pressure on the bottom of the bore may in a similar manner be expressed as follows:

$$P_0 = K_0 (3B)^{-1} \frac{fa}{\lambda} \left(\frac{\varpi}{p}\right)^{\frac{1}{4}} \frac{\Delta \varpi^{\frac{1}{2}}}{cu^{\frac{1}{2}}} x, \tag{94}$$

or simpler,

$$P_0 = K_0 (3B)^{-1} \frac{fa}{\lambda} \cdot \frac{\Delta \varpi^{\frac{3}{4}}}{pcu^{\frac{1}{2}}} x. \tag{95}$$

98. Monomial Formula for Velocity.—Equation (92) for the initial velocity contains the factor

$$f(x) = \frac{1}{2} x^{\frac{1}{2}} (3 - x).$$

Now, the preceding discussion leads us to suppose that in any system of artillery composed of different types of guns there must

be some value of x which, if used, will give results which do not differ materially from the truth. Therefore let us suppose that the binomial factor in question is capable of being replaced by a certain power of the variable x. On this hypothesis let

$$f(x) = Nx^n,$$

and (92) becomes

$$v = \frac{2}{3} A (3B)^{-\frac{1}{2}} N \left(\frac{fa}{\lambda}\right)^{\frac{1}{2}} \frac{\varpi^{\frac{3}{8}} \Delta^{\frac{1}{4}} c^{\frac{1}{4}} u^{\frac{1}{8}}}{p^{\frac{1}{2}}} x^n. \tag{96}$$

Replacing x by its value from (89), and putting, for brevity,

$$M = \frac{2}{3} A (3B)^{n-\frac{1}{2}} N,$$

we have the following formula:

$$v = M \left(\frac{fa}{\tau}\right)^{\frac{1}{2}} \left(\frac{\tau}{\lambda}\right)^{\frac{1}{2}-n} \frac{\varpi^{\frac{3}{8}} \Delta^{\frac{1}{4}} c^{\frac{1}{4}-n} u^{\frac{1}{8}+\frac{n}{2}}}{p^{\frac{1}{2}-\frac{n}{2}}}. \tag{97}$$

According to exhaustive experiments at Gavre it was found that with relatively quick powders the exponent of u was $\frac{1}{8}$, or that the velocity varied as the $\frac{1}{8}$ power of the travel of the projectile, but with relatively slower powders it became $\frac{1}{4}$, and increased as the powders used were relatively slower. Sarrau assumes $\frac{1}{8}$ and $\frac{1}{4}$ as the extreme values within which a monomial formula can be used, and adopts $\frac{3}{16}$, or the mean between $\frac{1}{8}$ and $\frac{1}{4}$, as that value which will best approximate to the normal conditions of practice. Adopting $\frac{3}{16}$ as the exponent of u in (97), n becomes $\frac{1}{8}$, and the formula for velocity is

$$v = M \left(\frac{fa}{\tau}\right)^{\frac{1}{2}} \left(\frac{\tau}{\lambda}\right)^{\frac{3}{8}} \frac{\varpi^{\frac{3}{8}} \Delta^{\frac{1}{4}} c^{\frac{1}{8}} u^{\frac{3}{16}}}{p^{\frac{7}{16}}}. \tag{98}$$

One firing with the standard powder is sufficient to determine M, and Sarrau finds by using the standard powder in the 24-cm. gun that $M = 322.45$, $\quad$ Log $M = 2.50846$,

the units being the foot, pound, and second.

If we adopt $\frac{1}{4}$ as the exponent of u, we get

$$v = M \left(\frac{fa}{\tau}\right)^{\frac{1}{2}} \left(\frac{\tau}{\lambda}\right)^{\frac{1}{4}} \frac{\varpi^{\frac{3}{8}} \Delta^{\frac{1}{4}} u^{\frac{1}{4}}}{p^{\frac{3}{8}}}, \tag{99}$$

which presents the curious feature, that with a slow powder and using the monomial formula, the velocity is independent of the calibre of the gun.

Formula (98) is that usually employed.

99. Limit of the Use of the Binomial Formula for Velocity.— Since the binomial formula contains two terms which increase together as τ decreases, the value τ_1 which makes v a maximum should be considered as fixing the extreme limit for the employment of the binomial formula. Consequently the binomial formula should not be used for powder which is quicker than the maximum. If we adopt $\frac{9}{11}$ as the superior limit of the modulus, according to Sarrau, and recollecting that the value of the subtractive term is $\frac{1}{3}$ for that value of τ giving a maximum velocity, we should cease to use the binomial formula when the value of the subtractive term is equal to or greater than $\frac{1}{3} \times \frac{9}{11} = .273$, about.

When the subtractive term becomes greater than .273, the monomial formula for quick powders will give the more accurate results.

100. Initial Velocities given by two Powders which give the same Maximum Pressure.— The monomial formula for velocity enables us to compare approximately the velocities for two powders giving the same maximum pressure.

We have

$$v = M\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\tau}{\lambda}\right)^{\frac{3}{8}} \frac{\varpi^{\frac{3}{8}}\Delta^{\frac{1}{4}}c^{\frac{1}{8}}u^{\frac{3}{16}}}{p^{\frac{7}{16}}}.$$

Take two powders having the characteristics

$$f,\ \tau,\ a,\ \lambda,$$
$$f',\ \tau',\ a',\ \lambda'.$$

In the same gun, under the same conditions of loading, the equality of pressures gives

$$\frac{fa}{\tau} = \frac{f'a'}{\tau'};$$

$$\therefore\ \frac{v'}{v} = \left(\frac{\lambda'\tau'}{\lambda'\tau}\right)^{\frac{3}{8}};$$

or, since $\frac{\tau'}{\tau} = \frac{f'a'}{fa}$,
$$\frac{v'}{v} = \left(\frac{f'a'\lambda}{fa\lambda'}\right)^{\frac{3}{8}}. \qquad (100)$$

If the mixture is the same, the influence of the form of the grain may thus be determined.

101. Maximum Pressures produced by different Powders giving the same Initial Velocity.—From the velocity formula we have for equality of velocities,

$$\left(\frac{fa}{\tau}\right)^{\frac{1}{2}}\left(\frac{\tau}{\lambda}\right)^{\frac{3}{8}}=\left(\frac{f'a'}{\tau'}\right)^{\frac{1}{2}}\left(\frac{\tau'}{\lambda'}\right)^{\frac{3}{8}},$$

and

$$\frac{\tau'}{\tau}=\left(\frac{f'a'}{fa}\right)^{4}\left(\frac{\lambda}{\lambda'}\right)^{3}.$$

The ratio of the pressures is

$$\frac{P_0'}{P_0}=\frac{f'a'\tau}{fa\tau'}. \tag{101}$$

Substituting the value of $\frac{\tau'}{\tau}$,

$$\frac{P_0'}{P_0}=\left(\frac{fa\lambda'}{f'a'\lambda}\right)^{3}. \tag{102}$$

102. To give an example, we shall compare the effects of two powders of the same make and differing in form and density of grain.

Take two powders, the grains of one being spheres and those of the other parallelopipeds. In the latter, let the base be a square, and the side of the base double the height. We have:

1st. For the sphere,

$$a=3,\ \lambda=1;$$

2d. For the parallelopiped,

$$a'=2,\ \lambda'=\tfrac{5}{8}.$$

We therefore have from (94) and (95),

$$\frac{v'}{v}=\left(\frac{16}{15}\right)^{\frac{3}{8}}=1.024,$$

$$\frac{P_0'}{P_0}=\left(\frac{15}{16}\right)^{3}=0.824.$$

By the substitution of the parallelopiped for the sphere, we may therefore increase the velocity by $\frac{1}{50}$, about, without changing the pressure, or diminish the pressure by about $\frac{1}{6}$ without changing the velocity.

In the first case, the ratio of the times of burning is

$$\frac{\tau}{\tau'}=\frac{3}{2};$$

and, in the second case,

$$\frac{\tau}{\tau'}=\left(\frac{3}{2}\right)^{4}\left(\frac{5}{8}\right)^{3}=1.236.$$

From the ratio of the times of combustion we can deduce the number of grains to a kilogram.

For the spherical powder, let

r be the mean radius of grain,
δ the density of the powder,
v the velocity of combustion,
N the number of grains to the kilogram.

We have

$$\tau = \frac{r}{v}, \quad r = \left(\frac{3}{4\pi\delta N}\right)^{\frac{1}{3}}.$$

Similarly for the second powder, calling δ', v', N' the homologous parts to δ, v, N, and a the height of the grain, we have

$$\tau' = \frac{a}{2v'}, \quad a = \left(\frac{1}{4\delta' N'}\right)^{\frac{1}{3}}.$$

The material of the two powders being the same, we have $v' = v$, $\delta' = \delta$, and consequently

$$\frac{\tau}{\tau'} = 2\left(\frac{3N'}{\pi N}\right)^{\frac{1}{3}}.$$

We deduce from this expression, and from the two values of $\frac{\tau}{\tau'}$ above, that,

1st. To obtain the same maximum pressure we must have

$$\frac{N'}{N} = 0.44 \ldots;$$

2d. To obtain the same velocity,

$$\frac{N'}{N} = 0.33 \ldots$$

Examples.

1. In a 6-inch gun loaded with spherical powder of $\delta = 1.79$ and 123 granules per pound, and $\varpi = 32$ and $p = 75.4$ pounds, the pressure was 31,000 pounds per square inch, and the velocity 2000 f. s. Find ϖ and p in a similarly loaded 8-inch gun; and, if it be required that the pressure in the latter gun shall be the same with the same powder also in the form of spheres, find the proper number of granules per pound. If the guns are similar, what will be the velocity in the 8-inch when loaded in the manner thus determined?

$\varpi' = 75.85$
$p' = 178.72$
$N' = 52$
$V' = 2000.$

2. With a certain pressure, a velocity of 2000 f. s. has been attained with granules of the form of spheres; what velocity would be attained, the pressure remaining the same, by the use of the same powder in the form of pierced cylinders in which the ratio of the least to the greatest dimension is $= \frac{1}{2}$? $V' = 2259$ f. s.

3. With a certain velocity, a pressure of 25,000 pounds has been reached in using spherical powder; what pressure would be reached, the velocity remaining the same, in using the powder last described in Example 8? $P' = 13{,}608$ lbs.

4. It being assumed that the available variation of the specific gravity of gunpowder is in practice from 1.70 to 1.85; determine the ratio of the pressures in a gun when loaded with powders having these specific gravities; the size of the grain will be assumed to vary so as to keep the time of burning constant, all other conditions will be identical in the two guns, and the densities of loading unity.

$$\frac{P}{P'} = 1.056.$$

CHAPTER XI.

PRESSURE CURVES.

103. In designing guns it is indispensable to know something about the pressure at other points along the bore of a gun than that at which the maximum pressure takes place. Two methods may be followed.

First, to consider the whole charge converted into gas at the instant of maximum pressure, and from that point to expand to the muzzle as the projectile moves along the bore, as steam expands in a cylinder. The pressure at different points along the bore may then be obtained by the formula deduced in Chapter IV, assuming that the density of loading, found by dividing the weight of the charge by the capacity in which the powder burns, when substituted in formula (33) will give the correct pressure.

The capacity in which the charge burns is evidently equal to the volume of the bore and powder chamber behind the projectile at any given point of the travel of the projectile. The foregoing method is mentioned because it has been and is still used in designing guns. But it is thought that the results are not accurate and are notably less than may be expected in practice except with very quick-burning powders. In the case of slow powders, the maximum pressure has been reduced, and the muzzle energy has been kept the same or has been increased in the same gun. It follows, then, that the pressure is comparatively high well down the bore.

Second, to deduce an expression for the pressure from the formula for velocity—and whenever practicable, this method is recommended as being very approximate to the truth. In detail the method is as follows:

104. The formula for muzzle velocity is

$$v = A\left(\frac{fa}{\tau}\right)^{\frac{1}{2}} \varpi u^{\frac{3}{8}} \left(\frac{\Delta}{pc}\right)^{\frac{1}{4}} \left(1 - B\frac{\lambda}{\tau}\frac{(pu)^{\frac{1}{2}}}{c}\right),$$

and for the same gun, and conditions of loading with a given powder may be written

$$v = A_1 u^{\frac{3}{8}} (1 - B_1 u^{\frac{1}{2}}), \tag{103}$$

in which
$$A_1 = a\varpi^{\frac{3}{8}}\left(\frac{\Delta}{pc}\right)^{\frac{1}{4}},$$
$$B_1 = \beta\,\frac{p^{\frac{1}{2}}}{c}.$$

Differentiating (103) with respect to v and u, and dividing by dt,
$$\frac{dv}{dt} = \left(\frac{3}{8}A_1 u^{-\frac{5}{8}} - \frac{7}{8}A_1 B_1 u^{-\frac{1}{8}}\right)\frac{du}{dt}.$$

But $\frac{du}{dt} = v$, and $\frac{dv}{dt} =$ acceleration of the projectile, $\therefore\ \frac{p}{g}\frac{dv}{dt} =$ pressure on the base of the projectile; therefore
$$P_1 = \frac{p}{g}\left(\frac{3}{8}A_1 u^{-\frac{5}{8}} - \frac{7}{8}A_1 B_1 u^{-\frac{1}{8}}\right)v, \tag{104}$$
in which P is the total pressure on the projectile's base. For pressure per unit area, per square inch for example, c being the calibre in inches, we have
$$P = \frac{4p}{\pi c^2}\left(\frac{3}{8}A_1 u^{-\frac{5}{8}} - \frac{7}{8}A_1 B_1 u^{-\frac{1}{8}}\right)v. \tag{105}$$

Evidently, if the quantity $u =$ the whole travel of the projectile, for which v is the corresponding velocity, the value of P is the pressure on the base of the projectile at the muzzle. For any other travel of the projectile, and corresponding velocity, P may be found, and thus by decreasing the value of u, and finding the corresponding value of v by the velocity formula, the pressure on the base of the projectile may be found along the chase of the gun.

105. Formula (105) may be put in another form for convenience in working, as follows:
$$P = \frac{D - Eu^{\frac{1}{2}}}{u^{\frac{5}{8}}}\,.\,v, \tag{106}$$
in which
$$D = \frac{3a\varpi^{\frac{3}{8}}\Delta^{\frac{1}{4}}p^{\frac{3}{4}}}{2\pi c^{\frac{9}{4}}},\quad E = \frac{7a\beta p^{\frac{5}{4}}\varpi^{\frac{3}{8}}\Delta^{\frac{1}{4}}}{8\pi c^{\frac{13}{4}}}.$$

106. It is not recommended to depend upon the values of P as thus deduced, for a travel of the projectile less than one-half the total travel to the muzzle, because the velocity formula is not reliable for a small travel of the projectile.

107. Near the point for which the pressure is a maximum, the form of the curve may be found by using the method described in Chapter VII, using the table for y_0.

108. Now it is known, as has been already stated, that the pressure on the walls of the gun is greater than the pressure on the base of the projectile; and if we adopt a coefficient determined by experiment, and multiply the pressure on the base of the projectile by this coefficient, we shall have several points of the pressure curve along the chase; we have also the form of the curve near the maximum, and it now remains to join these two portions by a fair curve. The coefficient which is recommended is **1.43**, on the assumption that the pressure on the base of the projectile is $\frac{7}{10}$ the pressure on the walls of the gun. With a pressure curve thus drawn, and any assumed factor of safety, the thickness of the walls of the gun naturally follows.

EXAMPLES.

1. Show from the equation in Example 12, Chapter VIII, that if we express the pressure in the gun (P) in pounds per square inch, the velocity in feet per second, and the travel in feet, we have

$$P = v \cdot \frac{619390 - 48000 u^{\frac{1}{2}}}{12948 u^{\frac{5}{8}}}.$$

2. From the equation in Example 7, show that the pressure in the 8-inch as the shot clears the muzzle, when loaded as defined in Example 5, is $P = 11654$.

3. Writing the formula for velocity for Dupont's O. P. powder in a 6-inch B. L. R.,

$$v = A_1 (\varpi u)^{\frac{3}{8}} \left(\frac{\Delta}{pc}\right)^{\frac{1}{4}} \left[1 - B\left(\frac{pu}{c}\right)^{\frac{1}{2}}\right],$$

in which $A_1 = 625$, $B_1 = .003$; derive a formula which will express v as a function of u, when $\varpi = 54$, $p = 100$, $c = .5$, $\Delta = 1.03$.

Ans. $v = 1056.8 u^{\frac{3}{8}} - 63.409 u^{\frac{7}{8}}$.

4. Find v, when the travel of the projectile is $12.21'$, with the data from Example 3. *Ans.* $v = 2135$ f. s.

5. From Example 3 derive a formula for P, the pressure per square inch on the shot's base for any point of the bore of the 6-inch B. L. R.

$$\textit{Ans.}\ P = \frac{43.529 - 6.0941 u^{\frac{1}{2}}}{u^{\frac{5}{8}}} v.$$

6. What is the pressure on the base of the projectile at the muzzle, the travel of the shot being 12.21′ and the muzzle velocity 2135 f. s., in the case of the 6-inch B. L. R. of Example 3? *Ans.* 9937 lbs.

7. In this same gun, what are the pressures on the projectile's base when the projectile has travelled 10 feet, 8 feet, and 6 feet respectively?

Ans. 11678 lbs.
13720 "
16474 "

CHAPTER XII.

THE ROTATION OF RIFLE PROJECTILES.

In smooth-bore guns spherical projectiles are always used; those of an elongated form, similar to what is in use in rifles, would not answer, because the direction of their longer axis could not be assured. Spherical projectiles, weight for weight, can carry a larger bursting charge than any other form, but from the possibility of making rifle projectiles longer, a larger bursting charge may be thrown from a rifle than from a smooth-bore gun of any fixed calibre. The increased stress upon the gun, however, soon fixes a limit in this direction, and projectiles are not usually more than 2½ to 3 calibres from base to point.

Much may be said as to the comparative advantages of rifles and smooth bores, but the advantage of the rifle appears to consist purely in the fact that its projectile leaves the bore in a known direction and with a known rotation. An elongated projectile of ordinary form in flight has an advantage over a spherical projectile of equal weight as respects retardation, as will be very evident by considering that the column of metal to be retarded is longer and the cross section less; but the same law operates when the projectile is acquiring velocity in the gun to increase the powder pressure; thus, so far as regards this question alone there is apparently no gain in the use of rifles.

The increase in accuracy, due to the cause already stated, is very great. The mean error, at ordinary ranges, of a rifled gun is not more than one-fourth or one-third that of a smooth bore.

A rifle has thus a substantial advantage for all ordinary purposes. For destroying structures which do not present much resistance to penetration, like earthworks, smooth bores may give better results, since they throw a larger bursting charge, other things being equal. The latter have evidently great advantage in ricochet fire also.

In order to keep elongated projectiles steady in flight, with their geometrical axes as nearly in the trajectory as possible, they are given a high rotary velocity about this axis. This is accomplished by cutting spiral grooves on the inside of the bore, or *rifling* it, as this process is called.

Rifling is either *uniform* or *increasing:* that is, the angle between the axis of the bore and the helical groove on its surface is either the same at all points, or else it increases as we pass from the seat of the shot to muzzle. This angle is called the angle of twist or of rifling. The reason that it never has its maximum value at the bottom of the bore is because the powder pressure is there greatest and thus the danger of bursting the gun at that point greatest; so that it is best to relieve the gun as much as possible. Rifling, then, is defined as either uniform or increasing. To define it completely, we must know the magnitude of the angle of rifling at every point. If the cylinder forming the bore be rolled out on a plane, and a line drawn on it parallel to the axis of the bore, evidently a uniform spiral will be a straight line, and an increasing one a curve.

Uniform Twist.—Suppose it is required to lay off a copying bar to use with the rifling machine described in the chapter on guns. The angle of rifling might be stated; or, as is more commonly the case, the angle might be defined by stating the *pitch* of the rifling (for the grooves in a gun merely constitute a screw of very long pitch). If, then, the rifling is to be uniform and the pitch n calibres, the shot is to turn over once in n calibres, and we lay off a line nd in length, and erect at its extremity a perpendicular πd in length; the hypothenuse on the two lines thus drawn is the developed spiral, and the angle of rifling is

$$\tan^{-1}\frac{\pi}{n}.$$

In practice, nd would be of inconvenient length; and, since the angle of rifling and the length of the rifled part of the bore are known, a better method of proceeding is obvious.

Increasing Twist.—Suppose that it is required to lay off a copying bar for a *uniformly increasing spiral*, its inclination being $\tan^{-1}\frac{\pi}{m}$ at the beginning, and $\tan^{-1}\frac{\pi}{n}$ at the muzzle. Recollecting that the cylinder of the bore is supposed to be developed on a plane, draw rectangular axes, that of x being along the bore, and assume

$$y = ax + bx^2$$

for the equation to the curve, since the inclination of this curve changes uniformly. Differentiating, we have

$$\frac{dy}{dx} = a + 2bx,$$

whence, calling l the length of the rifled part, we find

$$y = \frac{\pi}{m} x + \frac{\frac{\pi}{n} - \frac{\pi}{m}}{2l} x^2.$$

By means of this equation, any desired number of points can be found, and the driving edge of the copying bar made to the correct profile. If the equation to the developed groove is of the form $y = ax^{\frac{3}{2}}$, with the origin at a distance p from the beginning of rifling, and if the rifling begins with one turn in m calibres, and increases to one turn in n calibres in a distance l, we have

$$\frac{dy}{dx} = \frac{3}{2} ax^{\frac{1}{2}},$$

and

$$\frac{\pi}{m} = \frac{3}{2} ap^{\frac{1}{2}},$$

$$\frac{\pi}{n} = \frac{3}{2} a(p + l)^{\frac{1}{2}}.$$

From these two equations we may find p and l. The values are

$$p = \frac{n^2 l}{m^2 - n^2},$$

$$a = \frac{2}{3} \frac{\pi}{mn} \frac{(m^2 - n^2)^{\frac{1}{2}}}{l^{\frac{1}{2}}} x^{\frac{3}{2}}.$$

Example 1. The 6-inch B. L. R. is 193.53 inches long from breech to muzzle. The twist of rifling begins with one turn in 180 calibres, and, in a length of 134 inches along the axis of the bore, increases to one turn in 30 calibres, thence to the muzzle, a distance of 9.85 inches, the twist is uniform or $\frac{\pi}{30}$. Deduce equations to the developed groove which will allow a rifling bar to be constructed, and find the distance of the origin of the curve from the beginning of rifling; the form of the equation for increasing twist being $y^2 = ax^3$.

Ans. $y = .005945 x^{\frac{3}{2}}$,
or $y^2 = .00003535 x^3$.
$y = \frac{\pi}{30} x$.
$x_0 = 3.83''$.

Example 2. In a 10-inch gun the length of the rifled part of the bore is 118 inches, and the twist increases from one turn in 100 calibres at the beginning to one turn in 40 calibres at the muzzle.

At what point is the twist one turn in 60 calibres, the form of the developed groove being a common parabola?

Ans. 65.56 inches from the muzzle.

Angular Velocity of Rotation.—If ω is the angular velocity of the projectile, i. e. the number of unit angles of circular measure it revolves through in one second, we must have

$$\omega = 2\pi s,$$

if s is the number of revolutions made in a second. Also, if the distance along the bore the projectile moves in one revolution is nd, in which n is the number of calibres and d the calibre in inches, the whole distance moved in one second is snd in inches. But at the muzzle the velocity of translation is V f. s., hence at the muzzle the velocity of rotation is

$$\omega = \frac{24\pi V}{nd},$$

because $12V = snd$. The angular velocity then increases as the muzzle velocity increases, and decreases when n or d decreases.

Linear Velocity of Rotation.—The linear velocity of rotation is the product of the distance r in inches of the point considered, from the axis, and the angular velocity:

$$\text{linear velocity} = \omega r,$$

$$= \frac{2\pi Vr}{nd} \text{ feet per second,}$$

$$\text{or } \frac{24\pi Vr}{nd} \text{ inches per second.}$$

Example 1. Compare the angular velocities of rotation of a 9-inch projectile, the twist of rifling being $\frac{\pi}{45}$, with that of a 3-inch projectile, twist $\frac{\pi}{30}$, when the muzzle velocity of each is 1400 f. s.

Ans. $\omega_1 : \omega_2 :: 9 : 2$.

Example 2. What is the linear velocity of rotation of a point on the circumference of a 3-inch shell, the twist being $\frac{\pi}{30}$ and the M. V. 1390 f. s.? *Ans.* 145.6 f. s.

Example 3. With uniform twist, does a projectile rotate as rapidly when half way down the bore as at the muzzle?

The Rotational Energy.—It is obvious, and will hereafter be demonstrated, that the change in the direction of the axis of rotation

of a projectile due to any cause will be less as the velocity of rotation is greater. The complete investigation of this matter has not been made, and it is the practice to impart a rotation which has been found by experiment to be sufficient. It is generally admitted that, with projectiles whose length is 2½ to 3 times their calibre, a surface velocity of about 110 foot-seconds is enough to keep them steady (Article *Gunmaking*, *Enc. Brit.*, 9th edition).

This being settled, the *angular velocity*, which is the linear velocity of a point at unit distance from the axis, is fixed. If the muzzle velocity of the projectile be also fixed, as would be the case in practice, the angle of rifling at the muzzle is fixed, and is

$$\tan^{-1}\frac{\omega r}{V},$$

V being the muzzle velocity, ω the angular velocity, and r the radius of the projectile.

The rotational energy of the projectile is therefore

$$\frac{M}{2}(k\omega)^2, \text{ or } \frac{Mk^2}{2}.\omega^2;$$

where M is the mass and k the radius of gyration of the projectile. Now, this energy may be impressed in an indefinite number of ways, and it is important to the safety of the gun and projectile to impress it in the best manner possible. This will evidently be attained by cutting the groove so that the pressure causing rotation shall be constant as the projectile comes out of the bore. We have then, calling θ any angle rotated through in the bore,

$$Mk^2.\frac{d^2\theta}{dt^2} = \text{moment of rotating forces} = \text{a constant.}$$

Taking the origin of plane rectangular coordinates at the beginning of a groove at the bottom of the bore, the axis of X being parallel to the axis of the bore; we have $y = k\theta$, where k is a constant. Substituting in the equation above, we find

$$\frac{d^2y}{dt^2} = \text{a constant.}$$

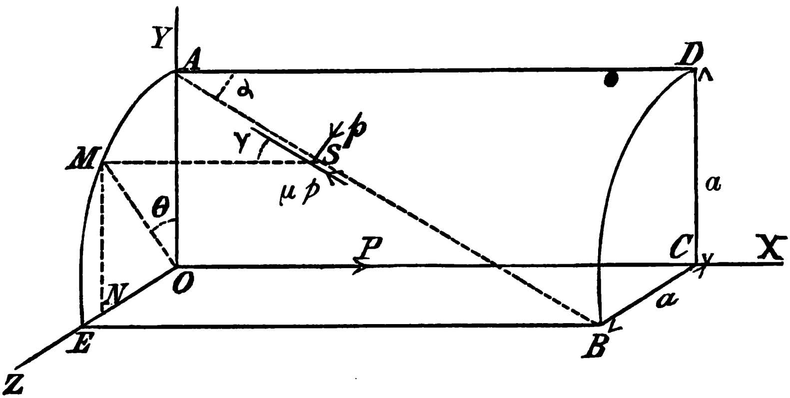
If the form of the function

$$\frac{d^2x}{dt^2} = f(x)$$

was known, we could, by integrating and eliminating t, find the correct form of rifling. If, for instance, the acceleration of translation is constant, the relation connecting x and y is linear.

The form of the function $f(x)$ is not known, and it probably varies sensibly in the actual conditions of manufacture of gunpowder. Thus, besides the straight line and parabola, already mentioned, curves given by $x^{\frac{5}{2}}=py$, $x^{\frac{7}{4}}=p'y$ have been used. The best practice is perhaps to use $x^{\frac{3}{2}}=py$ for the inner part of the bore, and a straight line coinciding with this at the point where they join for the muzzle part.

To Find the Pressure required to give Rotation to Projectiles.—In the figure, let $ADBE$ represent the surface of a quadrant of the bore of a gun; and AB, traced on it, the driving edge of a groove. Then, the muzzle being to the right, the rifling is right-



handed. We may assume that all the forces causing rotation act by one groove, and at the centre of pressure of all the forces which would really act along the bearing surfaces. Take the axes rectangular, the axis of x coinciding with the axis of the bore. Let P be the whole pressure on the base of the projectile, p the whole pressure causing rotation acting at the end of an arm a and at right angles to it, μ the coefficient of friction between the bearing surfaces in contact, and γ the angle between the groove and OX. Then the pressure causing motion along OX is

$$P-p(\sin\gamma+\mu\cos\gamma),$$

and that causing rotation is

$$pa(\cos\gamma-\mu\sin\gamma).$$

Let M be the mass of the projectile, k its radius of gyration, and

θ the angle turned through in any time t; then the equations of motion are

$$M\frac{d^2x}{dt^2} = P - p(\sin\gamma + \mu\cos\gamma), \qquad (1)$$

$$M\frac{d^2\theta}{dt^2} = \frac{pa(\cos\gamma - \mu\sin\gamma)}{k^2}. \qquad (2)$$

Uniform Twist.—In the figure, take OX as stated, and the plane of YZ through the origin of the grooves, with the axis of y passing through the groove considered. The coordinates of any point S of AB are

$$\left.\begin{aligned} x &= MS = \text{arc } AM \cot\alpha = a\theta . \cot\alpha, \\ y &= NM = a\cos\theta, \\ z &= ON = a\sin\theta. \end{aligned}\right\} \qquad (3)$$

From (1) and (2) we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{[P - p(\sin\gamma + \mu\cos\gamma)]k^2}{pa(\cos\gamma - \mu\sin\gamma)} \cdot \frac{d^2\theta}{dt^2} \\ &= \frac{[P\sec\gamma - p(\tan\gamma + \mu)]k^2}{pa(1 - \mu\tan\gamma)} \frac{d^2\theta}{dt^2} \end{aligned}\right\} \qquad (4)$$

From (3),

$$\sec\gamma = \frac{ds}{dx} = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dx} = \frac{\sqrt{1 + \cot^2\alpha}}{\cot\alpha} = \sec\alpha,$$

as we might have anticipated; also, $\tan\gamma = \tan\alpha$.

From the 1st of (3),

$$d^2x = a . \cot\alpha . d^2\theta.$$

Substituting this in (4), and writing α for γ, gives

$$a\cot\alpha = \frac{(P\sec\alpha - p\tan\alpha + \mu)k^2}{pa(1 - \mu\tan\alpha)}.$$

Solving for p and writing in the usual notation

$$\tan\alpha = \frac{\pi}{n},$$

we have

$$p = \frac{\pi k^2\sqrt{\pi^2 + n^2}}{na^2(n - \pi\mu) + \pi k^2(\pi + n\mu)} \cdot P. \qquad (5)$$

Example.—With an English 10-inch gun we have

$$n = 40,\ k = .312 \text{ ft.},\ a = .417 \text{ ft.},\ \mu = .164,$$

$$\therefore p = .04426\, P.$$

Increasing Twist.—In the case of an increasing twist, the solution will be more readily effected by properly choosing the origin. If the initial inclination of the groove is to be zero, and it is to increase

uniformly to some fixed value at the muzzle, we may keep the axes exactly as already fixed; and we shall then have, for the first of (3),

$$x^2 = c\,.\,a\theta\,,$$

where c is a constant to be determined. This will be apparent upon rolling out the surface $ADBE$, and recollecting that this is the equation of a parabola referred to a tangent at the vertex and diameter. If, however, the inclination is not to be zero at the origin of the groove, as is usually the case, we may still retain this simple form of the equation to the groove by moving the origin far enough back to admit of its use. Take then, as the equation to the groove,

$$x^2 = c\,.\,a\theta. \tag{6}$$

Differentiating twice, we have,

$$\frac{d^2\theta}{dt^2} = \frac{2}{ca}\left[\left(\frac{dx}{dt}\right)^2 + x\frac{d^2x}{dt^2}\right] = \frac{2}{ca}\left(v^2 + x\frac{d^2x}{dt^2}\right). \tag{7}$$

Substituting from (1) and (2) in (7), we have

$$\frac{pa\,(\cos\gamma - \mu\sin\gamma)}{k^2.\,M} = \frac{2}{ca}\left[v^2 + x\,.\,\frac{P - p\,(\sin\gamma + \mu\cos\gamma)}{M}\right] \tag{8}$$

From the 2d and 3d of (3) and from (6), we find, as before,

$$\sec\gamma = \frac{\sqrt{4x^2 + c^2}}{c},\ \text{and}\ \tan\gamma = \frac{2x}{c}.$$

Dividing (8) by $\cos\gamma$, and substituting these, we have,

$$p = \frac{2k^2\sqrt{4x^2 + c^2}\,(Mv^2 + Px)}{ca^2\,(c - 2\mu x) + 2k^2x\,(2x + \mu c)}. \tag{9}$$

To determine c, and the position of the origin, the latter giving the initial value of x in (9), the groove being a uniformly increasing one with inclinations $\tan^{-1}\frac{\pi}{m}$ and $\tan^{-1}\frac{\pi}{n}$. The equation, the origin of axes being at the beginning of the groove, is

$$y = \frac{\pi}{m}x + \frac{\frac{\pi}{n} - \frac{\pi}{m}}{2l}x^2.$$

Completing the square,

$$\left[x + \frac{\frac{\pi}{m}}{\frac{\frac{\pi}{n} - \frac{\pi}{m}}{l}}\right]^2 = \frac{2l}{\frac{\pi}{n} - \frac{\pi}{m}}\left[y + \frac{\left(\frac{\pi}{2m}\right)^2}{\frac{\frac{\pi}{n} - \frac{\pi}{m}}{2l}}\right];$$

whence it follows, by comparison with (6), that

$$c = \frac{2l}{\frac{\pi}{n} - \frac{\pi}{m}},$$

and that the initial value of x is

$$\frac{l \cdot \frac{\pi}{m}}{\frac{\pi}{n} - \frac{\pi}{m}} = \frac{nl}{m-n}.$$

Example.—Show, if the groove is to be a semi-cubical parabola, defined by the equation $x^{\frac{3}{2}} = c_1 \cdot a\theta$, that $\sec\gamma = \frac{\sqrt{4c_1^2 + 9x}}{2c_1}$; and that, if it is to have the initial and final inclinations $\tan^{-1}\frac{\pi}{m}$ and $\tan^{-1}\frac{\pi}{n}$, $c_1 = \frac{3\sqrt{l}}{2} \cdot \frac{mn}{\pi\sqrt{m^2 - n^2}}$, and the initial value of x is $\frac{ln^2}{m^2 - n^2}$.

The mean value of the pressure necessary to produce any known muzzle energy of rotation may be easily found as follows: If p be the whole pressure causing rotation and a its arm, then, using the same notation as above,

$$Mk^2 \frac{d^2\theta}{dt^2} = pa;$$

hence
$$Mk^2 \int_0^{\omega} \frac{2d\theta \cdot d^2\theta}{dt^2} = 2pa \int_0^{\alpha} d\theta,$$

where the limits of integration are corresponding values of the angular velocity and the angle the projectile has turned through.

Therefore
$$p = \frac{M(k\omega)^2}{2 \cdot a\alpha}.$$

Example.—Show that the mean value of the pressure necessary to produce the rotational muzzle energy of the 8-inch M. L. R. is about 15 tons. (For quantities needed in the calculation, see *Ordnance Manual: k^2* is about 7.07 inches; the head being solid and body hollow.)

To find the maximum safe pressure in the couple turning a projectile, with respect to the stress on the gun.—Let S be the elastic strength of the metal of which the gun is made to resist torsional shearing stress; R_2 and R_1 the external and internal radii of the gun at the point considered. Suppose that the gun has been twisted about its axis so that its external layer is just at the elastic

limit; then, since each layer is stretched through a length proportional to its radius r, the tangential pull resisting the rotating couple at any point will be

$$S\frac{r}{R_2};$$

the moment of this force is $$S\frac{r^2}{R_2},$$

and the elemental resistance to torsion is

$$2\pi r \,.\, S\frac{r^2}{R_2} \,.\, dr.$$

Since the pressure causing rotation acts with an arm $2R_1$, we have

$$p \,.\, 2R_1 = \int_{R_1}^{R_2} 2\pi r \,.\, S \,.\, \frac{r^2}{R_2}\, dr;$$

integrating, $$p \,.\, 2R_1 = \frac{2\pi S}{R_2} \,.\, \frac{R_2^4 - R_1^4}{4};$$

hence $$p = \frac{\pi S(R_2^4 - R_1^4)}{4 \,.\, R_2 R_1}.$$

Example.—If we take $S = 10$ tons, $R_2 = 4$, $R_1 = 3$, $p = 114$ tons, about.

To Find the Number of Turns per Second a Projectile may make without Danger of throwing off the Rifling Band, supposing it to be held on only by its Strength.—Let R be the outside radius of the band, t its thickness, l its length (parallel to the longer axis of the projectile), ρ its mass per unit volume, and T its strength (either elastic or ultimate).

The acceleration along the radius, at any point of radius r, is

$$\frac{v^2}{r} = \frac{(\omega r)^2}{r} = \omega^2 . r = (2\pi n)^2 . r,$$

n being the number of turns made per second. Let θ be the angle between any plane of rupture and that passing through the axis and the point considered. Then, calling δv the element of volume, the equation of equilibrium is

$$2tlT = 4\pi^2 n^2 r\rho \Sigma \sin\theta \,.\, \delta v.$$

If the mass is homogeneous, $\rho\delta v = l\rho \,.\, rdr \,.\, d\theta$; hence

$$ltT = 4\pi^2 n^2 \rho l \int_{R-t}^{R}\int_0^{\frac{\pi}{2}} r \sin\theta \,.\, dr \,.\, d\theta,$$

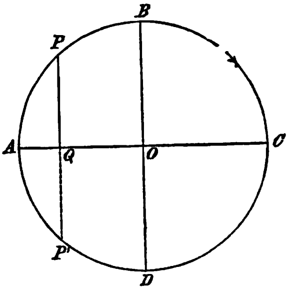
integrating, $$tT = 4\pi^2 n^2 \rho \,.\, \frac{R^3 - (R-t)^3}{3},$$

and $$n = \left(\frac{3T}{4\pi^2\rho(3R^2 - 3Rt + t^2)}\right)^{\frac{1}{2}}.$$

This formula is applicable only when t is small as compared with R.

Example.—If $R = 3''$, $t = \frac{1}{4}''$; then for cast brass, $T = 19{,}000$ pounds, and $\rho = .0098$: hence $n = 77$, about.

On the Steadying Effect of the Rotation of Projectiles.— Let the figure represent the intersection of an elongated projectile by a plane perpendicular to its axis of figure. Then, the rotation being as shown by the arrow, the motion is as it would appear to an observer behind a right-handed projectile. Consider the motion of a small mass μ at P in a very thin material circumference $ABCD$, impressed with the constant angular velocity ω_1 about O, and a constant angular velocity ω_2 about AC in such a direction as to cause the point B to rise from the plane of the paper. Then the problem presented is that usually met in the flight of projectiles, where the resultant of all the parallel forces of resistance of the air passes in front of the centre of gravity and tends to lift the point of the projectile.



The small mass at P is then impressed with two velocities; one of which is always situated in the turning plane $ABCD$, and the other is at right angles to it. The first is invariable in magnitude, but variable in direction; the second is variable in both direction and magnitude. The magnitude of the last velocity is $PQ.\omega_2$; and, since this changes as PQ changes, it is evident that all the elementary masses in the quadrant AB will, owing to their inertia, tend to lag behind, and all those in the quadrant BC will tend to advance out of, the plane $ABCD$. It will be found that these forces of inertia will, in each quadrant, tend to produce rotation of the plane $ABCD$ about an axis BD, in such a direction as to raise the point C from the paper. These forces, unless balanced, will then produce the rotation described. The change in the direction of the velocity $PQ.\omega_2$ will produce no external effect; since each force will be balanced by an equal and opposite one at the point P'. The elementary force caused by the change of $PQ.\omega_2$ may be evaluated as follows: its general expression, for a mass μ, is

$$\mu \cdot \frac{dv}{dt}.$$

In the case considered,

$$v = PQ \, . \, \omega_2 = R \, . \sin \alpha \, . \, \omega_2,$$

where R is the radius OA, and α the angle POA.

Also $$\alpha = \omega_1 t,$$

hence $$v = R\omega_2 \, . \sin \omega_1 t,$$

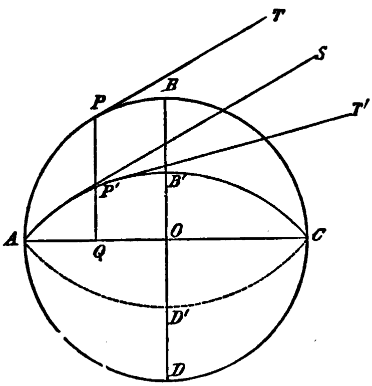
and $$\frac{dv}{dt} = R \, . \, \omega_1 \omega_2 \, . \cos \alpha \, .$$

The element of force due to this cause is therefore

$$\mu \, . \, R \, . \, \omega_1 \omega_2 \, . \cos \alpha,$$

and the sum of the moments of all these for the circumference may easily be obtained.

To determine the elementary force due to the change in the direction of the velocity $R\omega_1$:



In the figure, suppose $ABCD$ to have rotated through a very small angle into the position $AB'CD'$; then the direction of the velocity has changed from PT to $P'T'$. This change may be decomposed into two parts; one in the plane $ABCD$, and the other at right angles to it. The first will have no external effect, since it will be balanced by an equal and opposite force caused by an equal mass at the other end of the diameter. To find the magnitude and direction of the second component, draw $P'S$ parallel to PT, and we see that it is a centrifugal force in the plane $SP'T'$, and normal to $P'S$. Let $SP'T' = d\beta$; then for the force acting on a small mass μ, we have

$$\mu \frac{v^2}{\rho} = \mu v \, . \, \frac{d\beta}{dt},$$

since $$\rho = \frac{ds}{d\beta}.$$

We have $v = R\omega_1$, $POA = \alpha$, and $d\beta = SP'T' = PQP' . \cos \alpha = \omega_2 \, . \, dt \, . \cos \alpha$. Therefore

$$\mu v \frac{d\beta}{dt} = \mu \, . \, R\omega_1\omega_2 \cos \alpha \, .$$

Since the angle $SP'T'$ diminishes continuously as we pass from A to B, there changes sign and increases until we reach C, it is evident that all the elementary forces in the semi-circumference ABC will be superposed to produce rotation about BD; the same effect will be found in CDA. Therefore the elementary force tending to produce rotation of the circumference about BD, due to the coexistence of the angular velocities ω_1 and ω_2, is

$$2\mu \,.\, R\omega_1\omega_2 \cos a.$$

If we put $\mu_1 R \,.\, da$ for μ, the very small mass μ may be considered as an element of the circumference of mass μ_1 per unit length. The resultant moment acting is therefore

$$8\mu_1 R^3\omega_1\omega_2 \int_0^{\frac{\pi}{2}} \cos^2 a \,.\, da = 2\mu_1\pi R^3\omega_1\omega_2.$$

The moment of inertia of the circumference about its polar axis is

$$4\mu_1 \int_0^{\frac{\pi}{2}} r^2 . \, rd\theta = 2\pi .. \, \mu_1 R^3.$$

The above expression may therefore be written, calling A the polar moment,

$$A\omega_1\omega_2.$$

By similar processes, we may pass to the case of a very thin disk or to any solid of revolution, the above expression remaining always the same, and A being the polar moment of the body considered. We have then the following theorem, which enables us to determine the motion of a solid of revolution which is turning:

If a solid of revolution, whose moment of inertia about its geometrical axis is A, is rotating about that axis with a constant angular velocity ω_1, and is acted upon by a couple to change the direction of that axis; then, if, at any instant, ω_2 is the angular velocity about the axis of the couple, the resultant couple of the forces of inertia will be, at that instant, of magnitude $A\omega_1\omega_2$, and its axis will be at right angles to the geometrical axis and to the axis of the rotation ω_2.

Suppose that a projectile is moving in air as shown in the figure, the point being slightly above the trajectory, as would usually be the case; the resultant resistance, F, making a small angle with the direction of motion. If we conceive two equal and opposite forces, P, to be applied at the centre of gravity, we shall not alter the

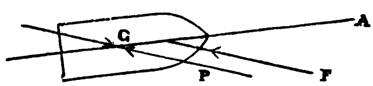


FIG. I.

circumstances of motion; and the forces acting may then be decomposed into a force P which retards the motion of translation, and a couple. The first may be left out of account, as it does not affect the matter in hand.

Now, at any instant, let θ be the angular velocity of the body about G in the plane PGA, and η that at right angles to this plane. Also, let ω_1 be its angular velocity about the axis AG; that which the gun impresses. Then the angular velocity θ will cause a couple of inertia $A\omega_1\theta$,
and the second a couple $A\omega_1\eta$;
the plane of the first being at right angles to the *moving* plane PGA and the second in this plane. There is also the couple caused by the force F in the plane PGA; calling this M, we have

$$\frac{d\theta}{dt} = \frac{M - A\omega_1\eta}{B}, \text{ and } \frac{d\eta}{dt} = \frac{A\omega_1\theta}{B}; \qquad (1)$$

where B is the moment of inertia about any axis in the plane through G (Fig. 1), at right angles to the geometrical axis. Dividing one of these by the other and integrating, we have, since θ and η are zero together,

$$\eta\,(2m - \eta) = \theta^2,$$

where

$$m = \frac{M}{A\omega_1}.$$

Drawing rectangular axes, we may trace the law connecting θ and η.

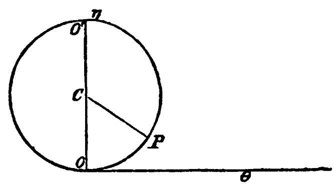


FIG. 2.

Let $OCP = \gamma$, then

$$\theta = m \sin\gamma, \text{ and } \eta = m\,(1 - \cos\gamma).$$

Substituting these values in the second of (1), we have, putting $b = \dfrac{B}{A\omega_1}$, $\dfrac{d\gamma}{dt} = \dfrac{1}{b}$; whence $\gamma = \dfrac{t}{b} + c = \dfrac{A}{B}\,\omega_1 t + c$.
Here γ is subject to the condition only that it must give $\theta = 0$ and $\eta = 0$, that is, $\sin\gamma = 0$ and $\cos\gamma = 1$, at the instant when the couple M begins to act. Reckoning the time from this instant, we have $c = 0$.

It is apparent from Fig. 1 that, when the couple M begins to act, the point of the projectile will rise; the effect of this rotation (θ) will be to produce a couple of moment $A\omega_1\theta$ at right angles to the moving plane PGA, and this couple will be found to turn the point of the projectile to the right, the observer being behind. If η be the angular velocity thus impressed, it will produce a couple of moment $A\omega_1\eta$ in the plane PGA, and of opposite sign to M. We shall have then, as has already been stated, the couples

$$M - A\omega_1\eta, \text{ and } A\omega_1\theta$$

in the plane PGA and at right angles to it. Consider the effect of the variable couple $M - A\omega_1\eta$ in the moving plane PA (Fig. 3).

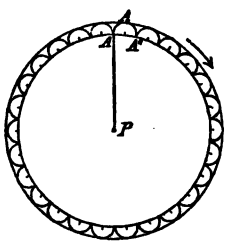


FIG. 3 (observer behind).

At the origin of time, when M is conceived to be applied, this will have the value M (see Fig. 2), and the velocity of the point A in the direction PA (Fig. 3) will increase. The acceleration of A caused by the couple

$$M - A\omega_1\eta$$

will decrease as η increases, pass through zero, and then become negative, if M is small enough (see Fig. 2). When the sum of all the negative accelerations equals the sum of all the positive ones, its value will be $-M$, and the point P will be at O' (Fig. 2). During the same interval, the couple $A\omega_1\theta$ will increase to a maximum, and then diminish to zero, its initial value. The velocity of the point A at right angles to the plane PA (Fig. 3) will therefore be a maximum at this time and its velocity in it will be zero; the point A will therefore describe some line AA' (Fig. 3).

The values of $\quad M - A\omega_1\eta$, and $A\omega_1\theta$,
when A is at A' are then $-M$ and 0. The point P (Fig. 2) while describing the second semi-circumference will cause the first of these to increase to $+M$; and, at that instant, the velocity of A in the plane PA will be zero. During the same interval, $A\omega_1\theta$ will always be negative, and the sum of all the negative impulses thus given to the point A (Fig. 3) will bring that point to rest; thus A will describe some line $A'A''$ during this interval. This line is identical in form, but different in position, from AA'. Thus when A reaches A'', the couple in the plane PA will be M, that at right angles to this plane will be zero, and the point will be at rest. Thus it is in the same condition as at A, and the same conditions will be repeated an indefi-

nite number of times if M and ω_1 do not change. The axis of the projectile will therefore describe in space a fluted cone whose intersection by a plane at right angles to the resultant resistance of the air is shown in Fig. 3.

The integrals of the expressions θdt and ηdt are, respectively, the angle described by the axis in the plane PGA, and the angular aperture of the developed cone. The first is

$$2bm\,(1 - \cos\gamma),$$

the maximum value of which is $2bm$. This is a very small quantity, and we may, therefore, as an approximation, neglect the motion of A in the plane PA (Fig. 3). Thus, we may assume

$$M - A\omega_1\eta = 0,$$

whence
$$\eta = \frac{M}{A\omega_1}.$$

This gives the angular velocity with which the axis will describe a mean right circular cone. We may, by this approximation, easily show the great advantage which an elongated projectile which is rotating has in steadiness over one which is not. At the end of the time dt the axis would have changed by the angle ηdt, about the diameter of the transverse section through the centre of gravity which is in the plane PGA. If there was no rotation ω_1, the same couple M would produce about the same axis an angular acceleration

$$\frac{M}{B}\,dt = \eta dt\,\frac{A\omega_1}{B}.$$

For example, in a 9-pounder gun, in which $\frac{A}{B} = \frac{1}{2.66}$, and $\omega_1 = 958$, the deviations undergone would be as 1 : 356.

The above discussion of rotation is taken from Théorie élémentaire des Phénomènes que presentent le Gyroscope, la Toupie, et le Projectile Oblong; par E. Jouffret.

Examples.

1. Show how to change (43) so as to take account of rotation and resistance to forcing.

Evidently (a) and (b), Chapter VI, will change: the second term of the latter becoming

$$\frac{m}{2}\cdot\left(\frac{du}{dt}\right)^2 + \frac{m}{2}\left(k\,\frac{d\theta}{dt}\right)^2,$$

where k is the radius of gyration of the projectile and $\frac{d\theta}{dt}$ its angular velocity. From equations (a) and (b), if F be the resistance to forcing, that is, the force which would be just sufficient to cause the shot to move steadily through the bore,

$$p\omega = m\frac{d^2u}{dt^2} + \frac{\sin\gamma + \mu\cos\gamma}{\cos\gamma - \mu\sin\gamma} \cdot \frac{mk^2}{a} \cdot \frac{d^2\theta}{dt^2} + F.$$

These results must then be substituted in ($39\frac{1}{2}$). Evidently, if the rifling be uniform, they will be greatly simplified.

2. Show that (43) satisfies the condition that each of its terms must be of the same dimensions.

Multiply through by $\frac{m}{u+u_0}$; and, for the dimensions of f, see (29).

3. A gun, its charge and projectile being conceived to be a system acted upon by no external forces, it is required to find the velocities of the gun and of the projectile after explosion of the charge.

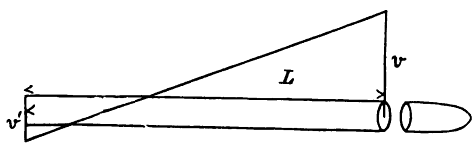
Let Q' be the part of the heat due to the combustion of the charge which, at the instant the shot's base clears the muzzle, has been utilized to produce translational energy of the shot, energy of recoil, and of motion of the charge. Then we have

$$2EQ' = mv^2 + m'v'^2 + \int w^2 . d\mu, \qquad (1)$$

where m and m', and v and v' are respectively the masses and velocities of the shot and gun; w is the variable velocity of the charge, and μ is its mass. We have also, since the external forces are zero,

$$mv - m'v' + \int w . d\mu = 0. \qquad (2)$$

If the mass of the charge be neglected, these give the solution at once. If, however, in the absence of more precise knowledge, we assume that the density of the charge at the instant considered is uniform; and that the velocity of the powder gas increases uniformly from its (negative) value at the bottom of the bore, where it is equal to the velocity of recoil, to its value at the base of the shot, where it is equal to the muzzle velocity of the shot; we may evaluate the integrals in (1) and (2) in the following manner: Calling L the whole length of the bore, the diagram shows that the section of gas which



is at the distance

$$\frac{L}{v+v'} \cdot v'$$

from the bottom of the bore will be at rest. We have then, taking x along the bore, its origin being at the point which is at rest,

$$w = \frac{v}{L\left(1 - \frac{v'}{v+v'}\right)} \cdot x = \frac{v+v'}{L} x;$$

also, $d\mu = \Delta_1 \omega \,.\, dx$, where Δ_1 is the uniform density of the gas and ω the cross section of the bore. The integral in (1) then becomes

$$\Delta_1 \omega \left(\frac{v+v'}{L}\right)^2 \int_{-\frac{L}{v+v'} \cdot v'}^{\frac{L}{v+v'} \cdot v} x^2 dx = \frac{\mu}{3} \cdot \frac{v^3 + v'^3}{v+v'},$$

since $\mu = \Delta_1 \omega \,.\, L$. The integral in (2) becomes

$$\Delta_1 \omega \frac{v+v'}{L} \int_{-\frac{L}{v+v'} \cdot v'}^{\frac{L}{v+v'} \cdot v} x dx = \frac{\mu}{2} \cdot (v - v').$$

Substituting these values in (1) and (2), we have

$$2EQ' = mv^2 \cdot \left(1 + \frac{\mu}{3m}\right) + m'v'^2 \left(1 + \frac{\mu}{3m'}\right) - \frac{\mu}{3} vv',$$

$$mv\left(1 + \frac{\mu}{2m}\right) - m'v'\left(1 + \frac{\mu}{2m'}\right) = 0.$$

From these equations the values of v and v' may be found.

This problem is taken from Saint-Robert's Mémoires Scientifiques, tome II, p. 191.

4. Show how to introduce a term taking account of the rotational muzzle energy in the result of Example 3.

Rigidly speaking, both the first and second terms of the second member of (1) would have to be altered by the addition of a term expressing the energy of rotation of projectile and gun. These would be of the form $m(k\omega)^2$; m, k, and ω being respectively the mass, radius of gyration, and angular velocity of the body considered. The angular velocities can then, the angle of rifling being known, be easily expressed in terms of v and v'.

5. If for simplicity we take $n = 2$ in the problem 8, Chapter VI, we find that if the maximum is reached as there stated, with a projectile weighing 300 lbs., the travel will be 6.58 inches. It is required to find the travel of the shot when we take account of the resistance to rotation, n being $= 2$.

Taking $\mu = .2$, $\tan\gamma = \frac{\pi}{30}$, $k = 3.744$ inches, we find from the result in Example 1, Chapter I, Part II,

$$p\omega = m \cdot \frac{d^2u}{dt^2}\left(1 + \frac{\pi}{30} \cdot \frac{k^2}{a^2} \cdot \frac{\tan\gamma + \mu}{1 - \mu\tan\gamma}\right);$$

whence

$$p\omega = m \cdot \frac{d^2u}{dt^2}(1 + .0183).$$

Exactly as before, we then find the travel to be 6.46 inches.

Since the maximum pressure depends very materially upon the position of the base of the shot at the instant of its occurrence, a comparison of this result with that of the last problem furnishes ground for the belief that the practice of making the inclination of rifling very small at the bottom of the bore, in order to lower the maximum pressure, is not sound.

6. Suppose that it requires a total pressure of 30 tons to force the projectile in Example 4 into the grooves; that is, suppose a force of 30 tons is just sufficient to cause the projectile to move steadily through the bore; find in this case, neglecting the resistance to rotation, how far the projectile will have moved when the maximum pressure is reached.

The shot's equation of motion in this case will be, taking the units as before,

$$p\omega = m\frac{d^2u}{dt^2} + 30.$$

From this, calling U the travel required, we find, $U = 5.74$ inches.